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Solving the Yang-Baxter-like matrix equation for diagonalizable coefficient matrix when A^3=-A

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Abstract— *Let A be a square matrix satisfying A^3=-A. We establish a method to find the solution of the Yang-Baxter matrix equation AXA=XAX in the case where the matrix A is a diagonalizable matrix with three different eigenvalues. Finally, we introduce a numerical example to demonstrate our method.*

***Keywords*— diagonalizable matrix, Yang-Baxter matrix equation, Jordan form, eigenvalue.**

I. INTRODUCTION

Let A be an $n \times n$ complex matrix. The quadratic matrix equation

$$AXA=XAX \tag{1.1}$$

is called the Yang-Baxter-like matrix equation.

Eq.(1.1) has its origin in the classical Yang-Baxter equation proposed in 1967 during the study of the multibody problem in nuclear physics [13]. The Yang-Baxter equation has applications in many fields of mathematics and physics, such as statistical mechanics, quantum field theory, differential equations, and knot theory.

Although much work has been done to find solutions to the Yang-Baxter equation in quantum field theory, no systematic study of Eq. (1.1) as a purely linear algebraic problem has been carried out. For one reason, finding the solution of Eq. (1.1) is equivalent to finding the solution of a system of n^2 quadratic equations with n^2 independent variables, and therefore, it is generally very difficult to find the solution.

Many of the studies carried out so far have been in finding a commutative solution of Eq. (1.1), i.e., a solution satisfying $AX=XA$ in [6,14-17]. In [3], all commutative solutions of Eq. (1.1) were found in the case of matrix A having some special Jordan structure, and in [4,5] the solution of matrix equations was studied in the case of diagonalizable and arbitrary nilpotent matrices. In [1], all solutions of matrix equation (1.1) are found for any square matrix with a general Jordan structure of matrix A.

Although, in connection with finding non-commutative solutions, limited studies have been carried out for some cases where the given matrix A is special. Methods for finding all solutions of rank 1 matrices [7], rank 2 matrices [8,9], and square equality matrices ($A^2=A$)[10] have been studied. Although in [12], all non-commutative solutions are found in the diagonalizable case where the given matrix has two different eigenvalues, and in [2], we have proposed a method to find all solutions of matrix equation (1.1) in the case of nilpotent matrices of rank 1 and 2.

In [11], the solution of matrix equation (1.1) satisfying $A^2=I$ and $A^3=A$ is found. However, we did not consider how to find the solution of the equation in the case of three nonzero eigenvalues with different matrices A.

We establish a method of finding the solution of Eq.(1.1) using characteristic polynomial and eigenvalues of A in the case where the given matrix A satisfying $A^3=-A$ has three different eigenvalues, based on the solution of Eq.(1.1) satisfying $A^2=-I$. Finally, numerical examples are presented to demonstrate the obtained results.

II. PRELIMINARY RESULTS

In this section, we give some lemmas and results for our further discussion.

Lemma 2.1. [11, Lemma 2.1] Let $A=\mathrm{diag}\{E, F\}$ be an $n\times n$ matrix and $\boldsymbol{E}$ an $s\times s$ matrix. Then the solution of Eq. (1.1) is

$$X=\begin{pmatrix} G & H \\ K & M \end{pmatrix}$$

Where $G\in C^{s\times s}, M\in C^{(n-s)\times(n-s)}$, H and K satisfies the following equation.

$$\begin{cases} EGE=GEG+HFK \\ EHF=GEH+HFM \\ FKE=KEG+MFK \\ FMF=KEH+MFM \end{cases} \tag{2.1}$$

In particular, if $F=0$, Eq. (2.1) can be expressed as

$$\begin{cases} EGE=GEG \\ GEH=0 \\ KEG=0 \\ KEH=0 \end{cases}$$

Lemma 2.2 [18. Lemma 2.1] Let $r\leqq n$ be two positive integers, and V and W are $n\times n$ and $r\times r$ matrices, respectively. If there exists an $n\times r$ matrix U satisfying $VQ=QW$ for any $n\times r$ matrix Q, then the following identity holds :

$$p_{V+QU^T}(\lambda)\cdot p_W(\lambda)\equiv p_V(\lambda)\cdot p_{W+U^TQ}(\lambda)$$

Let A be an n-type complex matrix. Since A^3=-A, the polynomial is an annihilating group of A, i.e., $p(A)$=0. Thus, the eigenvalues of A constitute a subset of the set A, and the minimal polynomial g(λ) of A is a single annihilator of A with the highest degree and the least degree of the term, and is a factor of $p(\lambda)$. Therefore, all eigenvalues of A are semi-simple, and then A is a diagonalizable matrix. Then there is a nonsingular matrix S such that $A=SJS^{-1}$ where $J = \mathrm{diag}(\lambda_1 I_{m_1}, \ldots, \lambda_r I_{m_r})$, $\lambda_j \in \{0, i, -i\}$, $j = 1, 2, \ldots, r$ and each I_{m_j} $(j = 1, 2, \ldots, r)$ represents $m_j \times m_j$ unit matrix. Then, if $Y=S^{-1}XS$, then it is clear that X is a solution of (1.1), if and only if Y is a solution of Eq.

$$JYJ=YJY \tag{2.2}$$

Also, X is a commutative solution if and only if Y is a commutative solution. Thus, to find the solution of Eq. (1.1), we solve Eq. (2.2). Then we divide the given matrix A into several cases and consider Eq. (1.1) for each case.

For minimal polynomials of matrix A, the solution is trivial in some cases. For example, if $g(\lambda)=\lambda$, then A=0, so all square matrices are solutions. If $g(\lambda)=\lambda\pm i$, then $A=\mp iI$. Then Eq. (1.1) has the form $\pm iX = (\pm iX)^2$. These cases are similar to the case of $A=I$ and only the coefficients are different. Therefore, only the following cases are considered.

Case 1. The minimal polynomial of the matrix A is $g(\lambda)=\lambda(\lambda-i)$.

Case 2. The minimal polynomial of the matrix A is $g(\lambda)=\lambda(\lambda+i)$.

Case 3. The minimal polynomial of the matrix A is $g(\lambda)=(\lambda-i)(\lambda+i)$.

Case 4. The minimal polynomial of the matrix A is $g(\lambda)=\lambda(\lambda-i)(\lambda+i)$.

We perform the analysis for these cases.

III. SOLUTION OF MATRIX EQUATION

3.1 Case 1. $A^2=iA$

Let A be an n-matrix with $A^2=iA$. Then A is a diagonalizable matrix and there exists a non-degenerate matrix S satisfying $A=SJS^{-1}$ for J=diag(iI_m,0), where I_m is the $m\times m$ -identity matrix and m is the rank of the matrix A. Then we denote the matrix Y as follows :

$$Y = \begin{pmatrix} G & H \\ K & M \end{pmatrix}$$

where G is an $m\times m$ matrix and M is a $(n-m)\times(n-m)$ matrix. Using Lemma 2.1, we have the following theorem.

Theorem 3.1. Let A be an $n\times n$ complex matrix whose rank is m and $A^2=iA$. Suppose that $A=S\mathrm{diag}(iI_m, 0)S^{-1}$ for any nonsingular matrix S. Then all solutions of the matrix equation (1.1) are given by

$$X = S\begin{pmatrix} G & H \\ K & M \end{pmatrix}S^{-1}.$$

If any $m\times m$ matrix U is expressed in the form $U=[U_1,U_2]$, its inverse matrix is

$$U^{-1} = \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{bmatrix}$$

where U_1 is an $m \times r$ matrix, $\hat{U}_1$ is an $r \times m$ matrix and $r \leq m$. And $G = iU_1 \hat{U}_1$, $H = U_2 Z_2$, $K = W_2 \hat{U}_2$, where any $(m-r)\times(n-m)$matrix Z_2 and $(n-m)\times(m-r)$matrix W_2 satisfy $W_2 Z_2 = 0$. Also, X is a commutative solution if and only if $W_2=0$, $Z_2=0$.

3.2 Case 2. $A^2=-iA$

Let A be an n-matrix with $A^2=-iA$. Then A is a diagonalizable matrix and there exists a nonsingular matrix S satisfying $A=SJS^{-1}$ for J=diag($-iI_m$,0). Then we denote the matrix Y as follow:

$$Y = \begin{pmatrix} G & H \\ K & M \end{pmatrix}$$

where G is an $m \times m$ matrix and M is a $(n-m)\times(n-m)$ matrix. Using Lemma 2.1, we have the following theorem.

Theorem 3.2. Let A be an $n \times n$ complex matrix whose rank is m and $A^2=-iA$. Suppose that $A=S\text{diag}(-iI_m, 0)S^{-1}$ for any nonsingular matrix S. Then all solutions of the matrix equation (1.1) are given by

$$X = S\begin{pmatrix} G & H \\ K & M \end{pmatrix}S^{-1}.$$

If any $m \times m$ matrix U is expressed in the form $U=[U_1,U_2]$, its inverse matrix is

$$U^{-1} = \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{bmatrix}$$

where U_1 is an $m \times r$ matrix, $\hat{U}_1$ is an $r \times m$ matrix and $r \leq m$. And $G=-iU_1 \hat{U}_1$, $H=U_2 C$, $K=B\hat{U}_2$, where any $(m-r)\times(n-m)$matrix C and $(n-m)\times(m-r)$matrix B satisfy $BC=0$. Also, X is a commutative solution if and only if $B=0$, $C=0$.

3.3 Case 3. $A^2=-I$

Let A be an n-matrix with $A^2=-iA$.

In this case A is a nonsingular matrix whose eigenvalues are i and $-i$. Let M be the multiplicity of the eigenvalue i, and J= diag $\{iI_m, -iI_{n-m}\}$ is the Jordan form of A.

To solve Eq. (2.2), let

$$Y = \begin{pmatrix} B & N \\ L & D \end{pmatrix}.$$

where B is an $m\times m$ matrix and D is a $(n\text{-}m)\times(n\text{-}m)$ matrix. From Eq. (2.1), Eq. (2.2) is equivalent to the following system of equations for the submatrices of Y:

$$\begin{cases} -B = iB^2 - iNL & ① \\ N = iBN - iND & ② \\ L = iKB - iDL & ③ \\ -D = -iD^2 + iLN & ④ \end{cases} \tag{3.1}$$

First, we find all commutative solutions of Eq. (2.2). If $JY=YJ$, since

$$\begin{pmatrix} iB & iN \\ -iL & -iD \end{pmatrix} = \begin{pmatrix} iB & -iN \\ iL & -iD \end{pmatrix}$$

,then $N=0$, $L=0$. Thus, we have the following result.

Theorem 3.3 Let A be an $n\times n$ complex matrix satisfying $A^2=-I$, and let m be the multiplicity of the eigenvalues i of the matrix. Then all the commutative solutions of Eq. (1.1) are $X = S\begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}S^{-1}$, where B and D are matrices with

$B^2 = iB,\ D^2 = -iD$.

Next, we find all non-commutative solutions of Eq. (2.2). From Eq. (3.1) ②,

$$BN = -iN + ND,\ \ ND = BN + iN$$

and, from ① and ④,

$$-iBN - iND = -B^2N + ND^2 = -B(BN) + (ND)D = iBN + iND\ ,$$

so

$$BN+ND=0.$$

Substituting the above expression into ②, we have $ND = \frac{i}{2}N,\ BN = -\frac{i}{2}N$. By ③

$$LB=-iL+DL,\ DL=iL+LB$$

and from ①,④,

$$-LB-DL=iLB^2-iD^2L=i(LB)B-iD(DL)=i(-iL+DL)-iD(iL+LB)=LB+DL$$

so

$$LB+DL=0$$

is derived. Substituting the above expression into ③, $DL = \frac{i}{2}L,\ LB = -\frac{i}{2}L$.

From the above result, we have

$$\begin{cases} B = -iB^2 + iNL \\ D = iD^2 - iLN \\ ND = \frac{i}{2}N \\ BN = -\frac{i}{2}N \\ DL = \frac{i}{2}L \\ LB = -\frac{i}{2}L \end{cases} \tag{3.2}$$

In Eq. (3.2), all nonzero columns of N and L are eigenvectors of B and D associated with eigenvalues $-\frac{i}{2}, \frac{i}{2}$, respectively, and all nonzero rows of N and L are left eigenvectors of D and B associated with eigenvalues $\frac{i}{2}, -\frac{i}{2}$, respectively.

Denote by $r(A)$ the rank of the matrix A. Then, to find all non-commutative solutions, consider the following lemma.

Lemma 3.1 If (B, N, L, D) is a solution of Eq. (3.2), then $r(N) = r(L)$.

***Proof*.** Let (B, N, L, D) satisfy Eq. (3.2). Using the first and fourth expressions of Eq. (3.2), we have $iNLN = iB^2N + BN = iB(BN) + BN = -\frac{3i}{4}N$, hence $NLN = -\frac{3}{4}N$.

Since $r(N) = r(NLN) \le r(NL) \le r(N)$, we have $NLN = -\frac{3}{4}N$.

Using the same method, from the second and fifth expressions, we have

$$iLNL = iD^2L - DL = iD(DL) - DL = -\frac{3i}{4}L,$$

so $LNL = -\frac{3}{4}L$. Hence, since $r(L) = r(LNL) \le r(NL) \le r(L)$, $r(L)= r(NL)$. Thus $r(N) = r(L)$. *End of **proof***.

By Lemma 3.1, $r(N) = r(L)= r$.

From the first and second expressions of Eq. (3.2), we have $NLN = -\frac{3}{4}N$ and $LNL = -\frac{3}{4}L$. Since $N\neq 0$, $L\neq 0$ and $BN = -\frac{i}{2}N$, $DL = \frac{i}{2}L$, the nonzero columns of N are eigenvectors corresponding to eigenvalues $-i/2$ of matrix B, and the nonzero columns of matrix L are eigenvectors corresponding to eigenvalues $i/2$ of matrix D. Then $-i/2$ and $i/2$ are the semisimple eigenvalues of B and D, respectively.

If the eigenvalue $-i/2$ of B is not a semisimple eigenvalue, there exists a nonzero vector $\boldsymbol{v}$ such that $u\equiv(B+i/2I)v\neq 0$ and $(B+i/2I)u = (B+i/2I)^2v = 0$.

The eigenvector u and the generalized eigenvector v must be linearly independent. If, for any $\gamma_1, \gamma_2\in C$, $\gamma_1 u+\gamma_2 v = \boldsymbol{0}$, $\gamma_1, \gamma_2\in C$, then

$$(B+\frac{i}{2}I)(\gamma_1 u+\gamma_2 v)=\gamma_1(B+\frac{i}{2}I)u+\gamma_2(B+\frac{i}{2}I)v=0$$

and so $\gamma_1=0$, $\gamma_2=0$.

$$NLv=(B^2-iB)v=(B+\frac{i}{2}I)^2 v-2i(B+\frac{i}{2}I)v-\frac{3}{4}v=-2iu-\frac{3}{4}v$$

Since $u=\frac{i}{2}NLv+\frac{3i}{8}v$, we have

$$0=(B+\frac{i}{2}I)^2 v=(B+\frac{i}{2}I)u=(B+\frac{i}{2}I)(\frac{i}{2}NLv+\frac{3i}{8}v)=\frac{3i}{8}u\neq 0.$$

This is a contradiction. Thus $-\frac{i}{2}$ is a semi-simple eigenvalue of B, and likewise $\frac{i}{2}$ is a semi-simple eigenvalue of D.

On the other hand, we apply Lemma 2.2 to the first expression of (3.2).

From

$$(iB)N=i(BN)=i(-\frac{i}{2}N)=N(\frac{1}{2}I_{n-m})$$

, we have

$$p_{iB+NL}(\lambda)p_{\frac{1}{2}I_{n-m}}(\lambda)\equiv p_{iB}(\lambda)p_{LN+\frac{1}{2}I_{n-m}}(\lambda).$$

Since $iB+NL=G^2$, $D^2+iD=LN$, by taking advantage of the fact that the eigenvalues of the square of the matrix are the square of the eigenvalues of the matrix, the above identity can be written as

$$\prod_{l=1}^{m}(\lambda-\lambda_l^2)(\lambda-\frac{1}{2})^{n-m}\equiv\prod_{j=1}^{m}(\lambda-i\lambda_j)\prod_{k=1}^{n-m}[\lambda-(\mu_k^2+i\mu_k+\frac{1}{2})]$$

where $\lambda_1,\lambda_2,\ldots,\lambda_m$ is the eigenvalues of B and $\mu_1,\mu_2,\ldots,\mu_{n-m}$ is the eigenvalue of D, $\lambda_l=-\frac{i}{2},\mu_l=\frac{i}{2}\quad(l=1,\ldots,r)$,respectively.

Dividing both sides by $(\lambda_l-\frac{1}{2})^r$, we have

$$\prod_{l=1}^{m}(\lambda-\lambda_l^2)(\lambda-\frac{1}{2})^{n-m-r}\equiv\prod_{j=r+1}^{m}(\lambda-i\lambda_j)\prod_{k=1}^{n-m}[\lambda-(\mu_k^2+i\mu_k+\frac{1}{2})]$$

and $\lambda_l^2=\mu_k^2+i\mu_k+\frac{1}{2}=-\frac{1}{4}\quad(l,k=1,\ldots,r)$, so

$$\prod_{l=r+1}^{m}(\lambda-\lambda_l^2)(\lambda-\frac{1}{2})^{n-m-r} \equiv \prod_{j=r+1}^{m}(\lambda-i\lambda_j)\prod_{k=r+1}^{n-m}[\lambda-(\mu_k^2+i\mu_k+\frac{1}{2})].$$

Thus, from the above identity, we have the following result. Thus

$$\lambda_l^2 = i\lambda_l \ , \ (l = r+1,\ldots,m) \ , \ \mu_k^2 = -i\mu_k \ ,(k = r+1,\ldots,n-m).$$

Hence, $\lambda_l = 0, i, (l = r+1,...,m)$ and $\mu_k = 0, -i, (k = r+1,...,n-m)$.

Next, we show that these eigenvalues are semi-simple.

If 0 is not a semi-simple eigenvalue of B, then there is a vector $v \neq 0$, $u \equiv iBv \neq 0$ and $B^2v=0$. From $-u=(B^2-iB)v=NLv$,

$$0=-Bu=BNLv=(i/2)v \neq 0$$

which is a contradiction.

If i is a semi-simple eigenvalue of B, then there is a vector $q \neq 0$ such that $p \equiv (B-iI)q \neq 0$ and $(B-iI)p=0$.

Since

$$NLq=(B^2-iB)q=(B-iI)^2q+i(B-iI)q=ip,$$
$$0=(B-iI)^2q=(B-iI)p=-i(B-iI)NLq=(-3/2)ip \neq 0$$

this is a contradiction. Thus 0 and i are semi-simple eigenvalues. In the same way, we can show that 0 and $-i$ are also semi-simple eigenvalues of the matrix D. Thus, matrices B and D are diagonalizable.

Summing up the above fact, we have the following theorem.

Theorem 3.4 Let A be an $n \times n$ complex matrix satisfying $A^2=-I$, and let k be the algebraic multiplicity of the eigenvalue i of the matrix. Then all non-commutative solutions of Eq. (1.1) are $X = S\begin{pmatrix} B & N \\ L & D \end{pmatrix}S^{-1}$.

Here B is any $k \times k$ diagonalizable matrix and D is any $(n-k)\times(n-k)$ diagonalizable matrix, satisfying the following properties.

① The rank of the nonzero matrices N and L are all r, $NLN = -\frac{3}{4}N$, and $LNL = -\frac{3}{4}L$.

② B and D have eigenvalues $-i/2$ and $i/2$ with multiplicity r, respectively.

③ The nonzero columns of N and the nonzero rows of L are the eigenvectors of B and the left eigenvectors associated with eigenvalue $-i/2$, respectively, and the nonzero rows of N and the nonzero columns of L are the eigenvectors of D and the left eigenvectors associated with eigenvalue $i/2$, respectively.

④ The other eigenvalues of B belong to $\{0, i\}$, and the other eigenvalues of D belong to $\{0, -i\}$.

3.4 Case 3. $A^3=-A$

Let A be an n-complex matrix with $A^2=-A$.

In this case, let A be a nonsingular matrix whose eigenvalues are i, $-i$, 0, and its rank be m. Let k be the multiplicity of the eigenvalue i, and $J=$ diag $\{iI_k, -iI_{m-k}, 0\}$ is the Jordan form of A.

Therefore, if Y is set to

$$Y = \begin{pmatrix} B & N & H_1 \\ L & D & H_2 \\ K_1 & K_2 & M \end{pmatrix},$$

then

$$JYJ = \begin{pmatrix} -B & N & 0 \\ L & -D & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$YJY = \begin{pmatrix} iB^2 - iNL & iBN - iND & iBH_1 - iNH_2 \\ iLB - iDL & iLN - iD^2 & iLH_1 - iDH_2 \\ iK_1B - iK_2L & iK_1N - iK_2D & iK_1H_1 - iK_2H_2 \end{pmatrix}.$$

From Eq. (2.2), the matrix M is any$(n\text{-}m)\times(n\text{-}m)$ matrix. First, we find all commutative solutions of Eq. (2.2). If $YJ=JY$, then

$$\begin{pmatrix} iB & -iN & 0 \\ iL & -iD & 0 \\ iK_1 & -iK_2 & 0 \end{pmatrix} = \begin{pmatrix} iB & iN & iH_1 \\ -iL & -iD & -iH_2 \\ 0 & 0 & 0 \end{pmatrix}$$

and N=0, H_1=0, L=0, H_2=0, K_1=0, K_2=0. Thus, every solution of Eq. (2.2) must satisfy the following equation :

$$B^2 = iB, \quad D^2 = -iD$$

Then we have the following result.

Theorem 3.5. Let A be an n-type complex matrix satisfying A^3=-A. Let the rank of A be m and the multiplicity of the eigenvalue i of A be k. Then all commutative solutions of Eq. (1.1) are obtained by the following expression :

$$X = S\,\mathrm{diag}\{B, D, M\}S^{-1}$$

where M is any $(n\text{-}m)\times(n\text{-}m)$ matrix, B is a $k\times k$ matrix satisfying $B^2 = iB$, and D is a $(m\text{-}k)\times(m\text{-}k)$ matrix satisfying $D^2 = -iD$.

Next, we find all non-commutative solutions of Eq. (1.1). Let $J = \mathrm{diag}\{T, 0\}$ be in Eq. (2.2). Here, $T = \mathrm{diag}\{iI_k, -iI_{m-k}\}$. Let Y be

$$Y = \begin{pmatrix} \hat{G} & H \\ K & M \end{pmatrix}$$

and $\hat{G}$ has the same size as *T*. Introducing Lemma 2.1, we have

$$\begin{cases} T\hat{G}T = \hat{G}T\hat{G} \\ T\hat{G}H = 0 \\ KT\hat{G} = 0 \\ KTH = 0 \end{cases} \tag{3.3}$$

and *M* is an arbitrary $(n-m)\times(n-m)$ matrix.

Next, for some special cases, we have the following theorem.

Theorem 3.6. Let *A* be an $n\times n$ complex matrix whose rank is *m* and satisfies $A^3=-A$, and let the multiplicity of eigenvalue *i* be *k*.

(1) If $\hat{G}=0$, then all solutions of Eq. (3.3) are $(0,H,K,M)$ and $KTH=0$. Also, if $H=0$, $K=0$, then all solutions are $(0,0,K,M)$ or $(0,H,0,M)$.

(2) If $H=0$, then all solutions of Eq. (3.3) are $(\hat{G},0,K,M)$, where $\hat{G}$ is the solution of the Young Baxter equation $T\hat{G}T=\hat{G}T\hat{G}$. And all the rows of *K* belong to the left null space of $T\hat{G}$. Also, if $K=0$, then all solutions are commutative.

(3) If $K=0$, then all solutions of Eq. (3.3) are $(\hat{G},H,0,M)$, where $\hat{G}$ is the solution of the Young Baxter equation $T\hat{G}T=\hat{G}T\hat{G}$. And all rows of *H* belong to the null space of $\hat{G}T$. Also, if $H=0$, then all solutions are commutative.

Next, let us solve Eq. (3.3) to find all non-commutative solutions of Eq. (2.2). Since the first equation of Eq. (3.3) is the Young Baxter matrix equation for the nonsingular matrix $T=\mathrm{diag}\{iI_k,-iI_{m-k}\}$ satisfying $T^2=-I_m$, its general solution is found from Theorem 3.4. Then, for all solutions $\hat{G}$ found, we solve the remaining three equations of Eq. (3.3) to find *H* and *K*. This implies that all solutions of Eq. (1.1) can be found from Lemma 2.1 to all solutions *Y* of matrix equation (2.2) for matrix $J=\mathrm{diag}\{T,0\}$ satisfying $T^2=-I$.

From Theorem 3.4, all solutions of the first equation of Eq. (3.3) are expressed as follows.

$$\hat{G}=\begin{pmatrix} B & N \\ L & D \end{pmatrix}$$

Here $\hat{G}$ is an arbitrary m-diagonalizable matrix and *M* is an arbitrary $(n-m)\times(n-m)$ diagonalizable matrix, satisfying the following properties.

① The rank of matrices N and L are all $r≧0$, and are $NLN = -\frac{3}{4}N$ and satisfy $LNL = -\frac{3}{4}L$.

② B and D have eigenvalues $-i/2$ and $i/2$ with multiplicity r, respectively.

③ The nonzero columns of N and the nonzero rows of L are the eigenvectors of B and the left eigenvectors associated with eigenvalue $-i/2$, respectively, and the nonzero rows of N and the nonzero columns of L are the eigenvectors of D and the left eigenvectors associated with eigenvalue $i/2$, respectively.

④ The other eigenvalues of B belong to $\{0, i\}$, and the other eigenvalues of D belong to $\{0, -i\}$.

⑤ A solution $\widehat{G}$ is commutative if and only if L=0, N=0, and in this case B and D are matrices with $B^2 = iB$, $D^2 = -iD$.

Let $\widehat{G}$ be any solution of the first equation of Eq. (3.3), and all four submatrices (B,N,L,D) from Eq. (3.3). Let us solve the remaining three equations for K and H. Since $T = \text{diag}\{iI_k, -iI_{m-k}\}$, we denote K and H as follows.

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \quad K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$$

Then the remaining three equations of Eq. (3.3) are :

$$\begin{pmatrix} iB & -iN \\ iL & -iD \end{pmatrix}\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} K_1 & K_2 \end{pmatrix}\begin{pmatrix} iB & iN \\ -iL & -iD \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} K_1 & K_2 \end{pmatrix}\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = 0.$$

Summarizing the above results, we have the following theorem.

Theorem 3.7. Let A be an $n×n$ complex matrix whose rank is m and satisfies $A^3=-A$, and let the multiplicity of eigenvalue i be k. Then all solutions of Eq. (1.1) are found as follows :

$$X = S\begin{pmatrix} B & N & H_1 \\ L & D & H_2 \\ K_1 & K_2 & M \end{pmatrix}S^{-1}$$

Here, the matrix of type $k×k$ B, the matrix of type $k×(m-k)$ N, $(m-k)×k$, $(m-k)×(m-k)$ matrix L and D satisfies Eq. (3.1), and is obtained by Theorem 3.3 and Theorem 3.4 when n is replaced by m. And any nonzero column vector $h=(h_1^T, h_2^T)^T$ of $m×(n-m)$ matrix $(H_1^T, H_2^T)^T$ and any nonzero row vector matrix $(H_1^T, H_2^T)^T$ of $(n-m)×m$ matrix $k=(k_1, k_2)$ are the eigenvectors and left eigenvectors of matrix

$$\begin{pmatrix} iB & -iN \\ iL & -iD \end{pmatrix}, \begin{pmatrix} iB & -iN \\ iL & -iD \end{pmatrix}$$

respectively, and $k_1\ h_1 = k_2\ h_2$ and M is the $(n\text{-}m)\times(n\text{-}m)$ matrix.

IV. NUMERICAL EXAMPLE

To demonstrate our results, we consider the following two examples.

Example 4.1. Let

$$A = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & -1 \\ 1 & 0 & 1 & -1 \end{pmatrix}.$$

Then $A^2 = -I$, its Jordan type is

$$\boldsymbol{J} = \text{diag}(i, i, -i, -i)$$

and

$$S = \begin{pmatrix} 1+i & -1 & 1-i & -1 \\ i & -1-i & -i & -1+i \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \text{ and } S^{-1} = \begin{pmatrix} 1+i & -1 & 1-i & -1 \\ i & -1-i & -i & -1+i \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

From Theorem 3.3, all commutative solutions of Eq. (1.1) are $X = S\text{diag}\{B, D\}S^{-1}$, where B and D are matrices with $B^2 = iB,\ D^2 = -iD$.

Next, we find the non-commutative solution of Eq. (1.1). We obtain

$$L = \begin{pmatrix} \frac{3\sqrt{3}}{2}i & -\frac{\sqrt{3}}{2}i \\ \frac{3\sqrt{3}}{2}i & -\frac{\sqrt{3}}{2}i \end{pmatrix},\ N = \begin{pmatrix} \frac{3\sqrt{3}}{2}i & -\sqrt{3}i \\ 3\sqrt{3}i & -2\sqrt{3}i \end{pmatrix},\ B = \begin{pmatrix} -\frac{7}{2}i & \frac{3}{2}i \\ -9i & 4i \end{pmatrix},\ D = \begin{pmatrix} \frac{7}{2}i & -3i \\ \frac{9}{2}i & -4i \end{pmatrix}.$$

Then, one non-commutative solution of Eq. (1.1) is

$$X = \begin{pmatrix} 1+i & -1 & 1-i & -1 \\ i & -1-i & -i & -1+i \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{7}{2}i & \frac{3}{2}i & \frac{3\sqrt{3}}{2}i & -\sqrt{3}i \\ -9i & 4i & 3\sqrt{3}i & -2\sqrt{3}i \\ \frac{3\sqrt{3}}{2}i & -\frac{\sqrt{3}}{2}i & \frac{7}{2}i & -3i \\ \frac{3\sqrt{3}}{2}i & -\frac{\sqrt{3}}{2}i & \frac{9}{2}i & -4i \end{pmatrix} \begin{pmatrix} -\frac{1}{2}i & 0 & -\frac{1}{2}i & \frac{1}{2}+\frac{1}{2}i \\ -\frac{1}{2}i & \frac{1}{2}i & \frac{1}{2} & \frac{1}{2}i \\ \frac{1}{2}i & 0 & \frac{1}{2}i & \frac{1}{2}-\frac{1}{2}i \\ \frac{1}{2}i & -\frac{1}{2}i & \frac{1}{2} & -\frac{1}{2}i \end{pmatrix}$$

$$=\begin{pmatrix} \frac{3}{2}+\frac{\sqrt{3}}{4}+(-\frac{3}{4}-\frac{3\sqrt{3}}{4})i & \frac{7}{4}+\frac{1}{2}\sqrt{3}+(\frac{3}{4}-\frac{3\sqrt{3}}{4})i & 1+\sqrt{3}+(-\frac{3}{4}-\sqrt{3})i & 2+3i-\frac{\sqrt{3}}{4} \\ \frac{11}{4}+(\frac{3}{2}+\frac{\sqrt{3}}{4})i & 4+\frac{3}{4}\sqrt{3}+(\frac{3}{4}+\frac{\sqrt{3}}{2})i & \frac{17}{2}+\frac{\sqrt{3}}{4}+(\frac{9}{4}+2\sqrt{3})i & -6+\frac{3\sqrt{3}}{4}+(\frac{3}{4}-\frac{5\sqrt{3}}{2})i \\ -\frac{11}{4} & -4-\frac{3\sqrt{3}}{4} & -\frac{27}{4}-\frac{3\sqrt{3}}{4}-\frac{5\sqrt{3}}{4}i & \frac{11}{4}+(-\frac{9}{4}+\frac{9\sqrt{3}}{4})i \\ -\frac{5}{4}+\frac{\sqrt{3}}{4} & -\frac{9}{4}-\frac{\sqrt{3}}{4} & -\frac{7}{2}+(-\frac{3}{4}-\frac{3\sqrt{3}}{4})i & \frac{5}{4}+\frac{3\sqrt{3}}{2}i-\frac{\sqrt{3}}{4} \end{pmatrix}$$

Example 4.2. Let

$$A=\begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}$$

Then $A^3=-A$, its Jordan type is

$$J=\mathrm{diag}(i,i,-i,-i,0)$$

and, $S=\begin{pmatrix} -\frac{1}{2}i & 1+i & \frac{1}{2}i & 1-i & 0 \\ \frac{1}{2}-\frac{1}{2}i & i & \frac{1}{2}+\frac{1}{2}i & -i & 0 \\ \frac{1}{2}i & 0 & -\frac{1}{2}i & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$ and $S^{-1}=\begin{pmatrix} -1 & 1 & -i & 1 & 0 \\ -\frac{1}{2}i & 0 & -\frac{1}{2}i & \frac{1}{2}+\frac{1}{2}i & 0 \\ -1 & 1 & i & 1 & 0 \\ \frac{1}{2}i & 0 & \frac{1}{2}i & \frac{1}{2}-\frac{1}{2}i & 0 \\ 2 & -2 & 0 & -2 & 1 \end{pmatrix}$.

From Theorem 3.3, every commutative solution of Eq. (1.1) is $X=S\mathrm{diag}\{B,D,m\}S^{-1}$, where B and D are matrices with $B^2=iB$, $D^2=-iD$ and m is an arbitrary number.

Next, we find the non-commutative solution of Eq. (1.1). From Theorem 3.7, all solutions of Eq. (1.1) are found as follows.

$$X=S\begin{pmatrix} B & N & H_1 \\ L & D & H_2 \\ K_1 & K_2 & m \end{pmatrix}S^{-1}$$

where m is an arbitrary number, B is a 2×2 diagonalizable matrix with one eigenvalue $-i/2$ and the other eigenvalue 0 or i, D is an $i/2$ and the other eigenvalue is a 2×2 diagonalizable matrix with 0 or $-i$, and $NLN=-\frac{3}{4}N$ and $LNL=-\frac{3}{4}L$ is satisfied. And any nonzero column vector $h=(h_1^T, h_2^T)^T$ of the 4×1 matrix $(H_1^T, H_2^T)^T$ and any nonzero row vector $k=(k_1, k_2)$ of the 1×4 matrix (K_1, K_2) are the eigenvectors and left eigenvectors of the matrix

$$\begin{pmatrix} iB & -iN \\ iL & -iD \end{pmatrix}, \begin{pmatrix} iB & -iN \\ iL & -iD \end{pmatrix}$$

respectively, and $k_1 h_1 = k_2 h_2$. Also (B,N,L,D) is obtained from Example 4.1.

V. CONCLUSION

We have found all solutions of Eq. (1.1) using characteristic polynomial and eigenvalues of A in the case where the given matrix A satisfying $A^3=-A$ has three different eigenvalues, based on the solution of Eq. (1.1) satisfying $A^2=-I$. The same idea and technique in this paper can be applied to find all solutions of the Yang-Baxter matrix equation when A satisfies the condition $A^k=-A$, $k\in N$. In the future, we will solve the problem of finding all solutions of the Yang-Baxter matrix equation when A has more general case.

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The authors declare that they have no competing interests.

AUTHORS' CONTRIBUTIONS

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