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Study on Numerical Approach Solution of the System of Two-dimensional Fredholm Integral Equations by using Bernstein Polynomial

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***Abstract*— *Integral equations are extensively used in many physical models appearing in the field of plasma physics, atmosphere–ocean dynamics, fluid mechanics, mathematical physics and many other disciplines of physics and engineering. In this paper, we establish new numerical technique for the solution of the system of two-dimensional Fredholm integral equations (2DFIEs) of both first and second kinds on any finite interval. Our method which is based on Bernstein polynomial reduces the system of 2DFIEs to an algebraic linear system, and they can be solved using any standard rule. We also present convergence analysis and stability analysis of the proposed technique.*
***Keywords*— Bernstein polynomial, convergence, Fredholm Integral Equations, algebraic linear system, finite interval**

I. INTRODUCTION

The mathematical form of physical models mostly lead toward FIEs. FIEs occur in various physical and engineering models such as, signal processing, linear forward modeling, mass distribution of polymers in polymeric melt etc.

Many problems in applied mathematics, engineering, mechanics, mathematical physics and many other fields can be transformed into the second-kind two-dimensional integral equations [3,2,1,4,5]. Integral equations also arise as representation formulas in the solutions of differential equations and Some other applications of these equations can be found in [6,7].

These equations appear in electromagnetic and electrodynamics, elasticity and dynamic contact, heat and mass transfer, fluid mechanic, acoustic, chemical and electrochemical process, molecular physics, population, medicine and in many other fields [8-13].

In recent decades, many techniques were presented by different authors for the solution of FIEs. Babolian et al. [14] applied the decomposition method to solve the linear FIEs of the second kind. Vahidi and Mokhtari [15] proposed the decomposition method for a system of linear FIEs of the second kind. They show that the Adomian decomposition method is equivalent to Picard's method.

Rashidinia and Zarebnia [16] described the convergence of the approximate solution of the FIEs in which numerical solution is obtained by means of Sinc-collocation method. This technique converts the system of integral equation into an explicit system of algebraic equation.

Taylor expansion method was presented by Maleknejad et al. [17] with smooth or weakly singular kernel to solve FIEs of the second kind. Moreover, the modified homotopy perturbation method is introduced by Javidi [18] to find the system of linear FIEs.

Khan et al. [19] introduced a novel computing multi-parametric homotopy approach for the system of linear FIEs. Actually, this was a modified method that forms an improved homotopy and contains three convergence control parameters.

Half-sweep arith-metic mean method was presented by Muthuvalu and Sulaiman [20] to solve FIEs based on composite trapezoidal rule. They examine the effectiveness of the Half-Sweep arithmetic mean method for solving dense linear systems. Khan et al. presented discretization technique for solving mixed Volterra–fredholm integral equation and 2D Volterra integral equations arising in mathematical physics [21,22,23].

The system of FIEs of both kinds has been taken, then reduces the equations to an algebraic linear system and can be solved using any standard rule. Convergence analysis of the proposed technique and some useful numerical results are presented so that the reader could understand this idea easily[24].

In view of the literature, no attempt has been made to solve system of FIEs of the first kind by using any technique. Hence, it is necessary to study the method finding the approximate solution of the system of 2DFIEs occurring in various applied problems.

The main thirst here is that a variety of applied problems have their natural mathematical setting as an integral equations, thus there have the advantage of usually simple method of solution. In this technique the desired accuracy can be obtained by increasing the degree of the Bernstein polynomial. As the increase of the degree increases the computational cost, so a new method is presented using Bernstein polynomial to solve 2DFIEs of first and second kind on arbitrary intervals [a; b] in which we can obtain the required accuracy at a lower degree of polynomial.

The system of 2DFIEs of the first kind is of the form

$$f_i(r,s)=\sum_{j=1}^{m}\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)z_j(x,y)dxdy,\quad i=1,\dots,m \qquad (1.1)$$

Where $z_j(r,s)$ are the unknown functions, β_{ij} are the constant parameters, $H_{ij}(r,s,x,y)$ and $f_i(r,s)$ are predetermined real-valued functions, and $z_j(r,s)=[z_1(r,s),\ z_2(r,s),\dots,z_m(r,s)]$ is the vector solution to be determined.

The system of 2DFIEs of the second kind is of the form

$$\sum_{j=1}^{m}u_{ij}(r,s)z_j(r,s)=f_i(r,s)+\sum_{j=1}^{m}\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)z_j(x,y)dxdy,\quad i=1,\dots,m \qquad (1.2)$$

where $z_j(r,s)$ are the unknown functions, β_{ij} are constant parameters, $H_{ij}(r,s,x,y)$, $f_i(r,s)$ and $u_{ij}(r,s)$ are predetermined real-valued functions, and $z_j(r,s)=[z_1(r,s),\ z_2(r,s),\dots,z_m(r,s)]$ is the vector solution to be calculated.

The aim of this work is to use Bernstein polynomials for solving systems of FIEs. This paper consists of the following sections:

Section 2 describes some results and the basic concept of 2-dimensional Bernstein polynomials. In Section 3, the general method is explained. Section 4 shows the convergence analysis of the given technique and Hyers-Ulam stability of the proposed numerical technique. Finally, Section 5 contains conclusion and further work.

II. PRELIMINARY RESULTS

The Bernstein approximation $B_{n,k}(z)$ of a function $z_j:[a,b]\times[c,d]\to R$ is given as the following polynomial

$$B_{n,k}(z_j(r,s))=\sum_{p=0}^{n}\sum_{q=0}^{k}z_j(\frac{p}{n},\frac{q}{k})P_{n,k,p,q}(r,s) \tag{2.1}$$

where for $p=0,\ldots,n$ and $q=0,\ldots,k,$

$z_j^{p,q}=z_j(r_p,s_q),\ r_p=a+\dfrac{b-a}{n}p, s_q=c+\dfrac{d-c}{k}q$ and

$$P_{n,k,p,q}(r,s)=\frac{\begin{pmatrix}n\\p\end{pmatrix}\begin{pmatrix}k\\q\end{pmatrix}}{(b-a)^n(d-c)^k}(r-a)^p(b-r)^{n-p}(s-c)^q(d-s)^{k-q} \tag{2.2}$$

is the two-dimensional polynomial of degree (n, k). We are able to write

$P_{n,k,p,q}(r,s)=P_{n,p}(r)P_{k,q}(s)$ where

$$P_{n,p}(r)=\frac{\begin{pmatrix}n\\p\end{pmatrix}}{(b-a)^n}(r-a)^p(b-r)^{n-p},\ \ p=0,\ldots,n \tag{2.3}$$

are one-dimensional Bernstein polynomials which have the property $\sum_{p=0}^{n}P_{n,p}(r)=1$. The following lemma shows some properties of the one-dimensional Bernstein polynomial.

lemma 2.1. ([25]) If $P_{n,p}(r)$ be defined by (2.3), then

$$\sum_{p=0}^{n}i^kP_{n,p}(r)=\sum_{j=1}^{k}t_j\alpha_j(r), \tag{2.4}$$

where $t_1=1$ and for $j=2,3,\ldots,k$:

$$t_j=\sum_{n=2}^{j}\frac{(-1)^{j-n}(n^{k-1}-1)}{(n-1)!(j-n)!} \tag{2.5}$$

and for $j=2,3,\ldots,k$:

$$\alpha_j(r)=\frac{m(m-1)\ldots(m-j+1)}{(b-a)^j}(r-a)^j \tag{2.6}$$

lemma 2.2. ([25]) The Bernstein polynomials have the following properties:

i) $\sum_{p=0}^{n}(r_p - r)P_{n,p}(r) = 0$

ii) $\sum_{p=0}^{n}(r_p - r)^2 P_{n,p}(r) = \frac{1}{n}(r-a)(b-r)$

iii) $\sum_{p=0}^{n}(r_p - r)^3 P_{n,p}(r) = \frac{1}{n^2}(r-a)(b-r)(\frac{a+b}{2} - r)$

We show the following theorems about uniform convergence and error bound of the Bernstein approximation (2.1) for $z(r, s)$.

Theorem 2.1.([25]) For any function $z_j(r,s) \in C([a,b]\times[c,d])$ and any $\varepsilon > 0$, there exist a Bernstein polynomial approach sequence $\{B_{n,k}(z_j(r,s)),\ n,k = 1,2,...\}$ to the function $z_j(r,s)$ such that $\|B_{n,k}(z_j(r,s)) - z_j(r,s)\| < \varepsilon$. If $M = \|z_j(r,s)\|$, then then n, k have to satisfy $n > \frac{2M(b-a)^2}{\varepsilon\delta_1^2}$, $k > \frac{2M(d-c)^2}{\varepsilon\delta_2^2}$. Hence the sequence $\{B_{n,k}(z_j(r,s)),\ n,k = 1,2,...\}$ converges uniformly to $z_j(r,s)$.

Theorem 2.2. ([25]) If $z_j(r,s)$ is bounded on $[a,b]\times[c,d]$, has continues third order derivatives in $[a,b]\times[c,d]$, and $n = hk(h \in Q)$, then

$$\lim_{k\to\infty} k[B_{n,k}(z_j(r,s)) - z_j(r,s)] = \frac{1}{2h}(r-a)(b-r)z''_{j,rr}(r,s) + \frac{1}{2}(s-c)(d-s)z''_{j,ss}(r,s)$$

Theorem 2.3. If $z_j(r,s) \in C^2([a,b]\times[c,d]),\ j = 1,2,...,m,$ and $\|\cdot\|$ be the maximum norm on $[a,b]\times[c,d]$, then the error bound is

$$\|B_{n,k}(z_j(r,s)) - z_j(r,s)\| \le \frac{(b-a)^2}{8n}\|z''_{j,rr}(r,s)\| + \frac{(d-c)^2}{8k}\|z''_{j,ss}(r,s)\|. \quad (2.7)$$

Proof. From theorem 2.2,

$$\lim_{k\to\infty} k[B_{n,k}(z_j(r,s)) - z_j(r,s)] = \frac{1}{2h}(r-a)(b-r)z''_{j,rr}(r,s) + \frac{1}{2}(s-c)(d-s)z''_{j,ss}(r,s).$$

Hence

$$B_{n,k}(z_j(r,s)) - z_j(r,s) = \frac{1}{2hk}(r-a)(b-r)z''_{j,rr}(r,s) + + \frac{1}{2k}(s-c)(d-s)z''_{j,ss}(r,s) + o(\frac{1}{k}).$$

Take the maximum norm on both sides. Then

$$\left\| B_{n,k}(z_j(r,s)) - z_j(r,s) \right\| \le \frac{(b-a)^2}{8n} \left\| z''_{j,rr}(r,s) \right\| + \frac{(d-c)^2}{8k} \left\| z''_{j,ss}(r,s) \right\|. \quad \square$$

III. Numerical Approach Solution of the System of 2dfies by using Bernstein Polynomial

In this section, Bernstein basis functions are used to find numerical solutions of the system of 2DFIEs for first and second kinds. In the proposed technique, a discretization process is presented for both cases.

3.1. Numerical Approach Solution of the System of 2DFIEs of first kind

To obtain the numerical solution of the system of 2DFIEs of first kind (1.1),we will replace the unknown functions $z_j(x,y)$ with Bernstein basis function of degree (n, k), defined in (2.1) . Then the equation (1.1) becomes,

$$f_i(r,s) = \sum_{j=1}^{m} \sum_{p=0}^{n} \sum_{q=0}^{k} \beta_{ij} z_j^{p,q} \gamma_{pq} \int_c^d \int_a^b H_{ij}(r,s,x,y) A_{pq}(x,y) dxdy. \tag{3.1}$$

where $A_{pq}(x,y) = (x-a)^p (b-x)^{n-p} (y-c)^q (d-y)^{k-q}$ and $\gamma_{pq} = \dfrac{\binom{n}{p}\binom{k}{q}}{(b-a)^n (d-c)^k}$.

In order to calculate the values of $z_j^{p,q}$, $j = 1,2,\dots,m$, $p = 0,1,\dots,n$, $q = 0,1,\dots,k$, r and s are replaced with $r_u = a + \dfrac{(b-a)}{n} u + \varepsilon$, $u = 0,1,\dots,n-1$, $r_n = b - \varepsilon$ and $s_v = c + \dfrac{(d-c)}{k} v + \varepsilon$, $v = 0,1,\dots,k-1$, $s_k = d - \varepsilon$, respectively, where $0 < \varepsilon < 1$.

We can choose any other distinct nodes in [a, b] and [c, d] except singular values of our integral equation. The following linear equations system for $z_j^{p,q}$ is obtained

$$\sum_{j=1}^{m} \sum_{p=0}^{n} \sum_{q=0}^{k} c_{i.j,(u,v)}^{p,q} z_j^{p,q} = f_i(r_u, s_v), \; j = 1,2,\dots,m, \; u = 0,1,\dots,n, \; v = 0,1,\dots,k, \tag{3.2}$$

where.

$$c_{i.j,(u,v)}^{p,q} = \frac{\binom{n}{p}\binom{k}{q}}{(b-a)^n (d-c)^k} \beta_{ij} \int_c^d \int_a^b H_{ij}(r_u, s_v, x, y)(x-a)^p (b-x)^{n-p} (y-c)^q (d-y)^{k-q} dxdy. \tag{3.3}$$

The equation (3.2) is the system of $m(n+1)(k+1)$ linear equations with $m(n+1)(k+1)$ unknowns $z_j^{p,q}$, $j = 1,2,\dots,m$, $p = 0,1,\dots,n$, $q = 0,1,\dots,k$. So writing to the matrix form is as following.

$$CZ = F \tag{3.4}$$

where

$$C=\begin{bmatrix} C'_{1,1} & C'_{1,2} & \cdots & C'_{1,m} \\ C'_{2,1} & C'_{2,2} & \cdots & C'_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ C'_{m,1} & C'_{m,2} & \cdots & C'_{m,m} \end{bmatrix}, \qquad Z=\begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_m \end{bmatrix}, \qquad F=\begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_m \end{bmatrix}$$

and $C'_{i,j}$ are $(n+1)(k+1)$ matrix, Z_j, F_j are $(n+1)(k+1)$ column vectors, respectively, that is

$$C'_{i,j}=\begin{bmatrix} c^{0,0}_{i.j,(0,0)} & c^{0,1}_{i.j,(0,0)} & \cdots & c^{n,k}_{i.j,(0,0)} \\ c^{0,0}_{i.j,(0,1)} & c^{0,1}_{i.j,(0,1)} & \cdots & c^{n,k}_{i.j,(0,1)} \\ \vdots & \vdots & \ddots & \vdots \\ c^{0,0}_{i.j,(n,k)} & c^{0,1}_{i.j,(n,k)} & \cdots & c^{n,k}_{i.j,(n,k)} \end{bmatrix}, \quad Z_j=\begin{bmatrix} Z^{0,0}_j \\ Z^{0,1}_j \\ \vdots \\ Z^{n,k}_j \end{bmatrix}, \quad F_j=\begin{bmatrix} f_j(r_0,s_0) \\ f_j(r_0,s_1) \\ \vdots \\ f_j(r_n,s_k) \end{bmatrix}$$

It is therefore possible to solve system (3.4) by using a standard rule to obtain the unknowns $z_j^{p,q}$ and by using these unknowns in (2.1), we have $B_{n,k}(z_j^{p,q}(r_u,s_v))$, which is the solution of the system of 2DFIEs of the first kind.

3.2. Numerical Approach Solution of the System of 2DFIEs of second kind

To find the solution of the system of 2DFIEs (1.2), we replace the unknown function $z_j(r,s)$ and $z_j(x,y)$ by Bernstein basis function of degree (n, k) defined in (2.1).

Then the equation (1.2) becomes

$$\sum_{j=1}^{m}\sum_{p=0}^{n}\sum_{q=0}^{k} u_{ij}(r,s) z_j^{p,q}\gamma_{pq}A_{pq}(r,s)=f_i(r,s)+ + \sum_{j=1}^{m}\sum_{p=0}^{n}\sum_{q=0}^{k}\beta_{ij} z_j^{p,q}\gamma_{pq}\int_c^d\int_a^b H_{ij}(r,s,x,y)A_{pq}(x,y)dxdy, \quad i=1,\ldots,m \tag{3.5}$$

This might be represented

$$f_i(r,s)=\sum_{j=1}^{m}\sum_{p=0}^{n}\sum_{q=0}^{k} z_j^{p,q}\gamma_{pq}\Big[u_{ij}(r,s)A_{pq}(r,s)- -\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)A_{pq}(x,y)dxdy\Big], \quad i=1,\ldots,m \tag{3.6}$$

where $A_{pq}(x,y)=(x-a)^p(b-x)^{n-p}(y-c)^q(d-y)^{k-q}$, $\gamma_{pq}=\dfrac{\binom{n}{p}\binom{k}{q}}{(b-a)^n(d-c)^k}$.

To find $z_j^{p,q},\ j=1,2,\ldots,m,\ p=0,1,\ldots,n,\ q=0,1,\ldots,k$, the above equation can be written as a linear system of equations by replacing r with $r_u = a + \frac{(b-a)}{n}u + \varepsilon,\ u = 0,1,\dots,n-1,\ r_n = b - \varepsilon$ and s with $s_v = c + \frac{(d-c)}{k}v + \varepsilon,\ v = 0,1,\dots,k-1,\ s_k = d - \varepsilon$. Where $0<\varepsilon<1$. Similarly with the first kind, we can choose any other distinct nodes in [a, b] and [c, d] except singular values of our integral equation. Then the linear equations system for $z_j^{p,q}$ is obtained as following.

$$\sum_{j=1}^{m}\sum_{p=0}^{n}\sum_{q=0}^{k} d_{i.j,(u,v)}^{p,q} z_j^{p,q} = f_i(r_u, s_v),\ \ j = 1,2,\dots,m,\ \ u = 0,1,\dots,n,\ \ v = 0,1,\dots,k \tag{3.7}$$

where

$$d_{i.j,(u,v)}^{p,q} = \frac{\binom{n}{p}\binom{k}{q}}{(b-a)^n (d-c)^k}\Big[u_{ij}(r_u, s_v)(r_u - a)^p (b - r_u)^{n-p} (s_v - c)^q (d - s_v)^{k-q} - \\ - \beta_{ij}\int_c^d\int_a^b H_{ij}(r_u, s_v, x, y)(x-a)^p (b-x)^{n-p} (y-c)^q (d-y)^{k-q}\,dxdy\Big] \tag{3.8}$$

This equation is the system of $m(n+1)(k+1)$ linear equations with $m(n+1)(k+1)$ unknowns $z_j^{p,q},\ j = 1,2,\dots,m,\ p = 0,1,\dots,n,\ q = 0,1,\dots,k$. So writing to the matrix form is as following.

$$DZ = E \tag{3.9}$$

where

$$D = \begin{bmatrix} D'_{1,1} & D'_{1,2} & \cdots & D'_{1,m} \\ D'_{2,1} & D'_{2,2} & \cdots & D'_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ D'_{m,1} & D'_{m,2} & \cdots & D'_{m,m} \end{bmatrix},\quad Z = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_m \end{bmatrix},\quad E = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{bmatrix}$$

and $D'_{i,j}$ are (n+1)(k+1) matrix, Z_j, E_j are (n+1)(k+1) column vectors, respectively, that is,

$$D'_{i,j} = \begin{bmatrix} d_{i.j,(0,0)}^{0,0} & d_{i.j,(0,0)}^{0,1} & \cdots & d_{i.j,(0,0)}^{n,k} \\ d_{i.j,(0,1)}^{0,0} & d_{i.j,(0,1)}^{0,1} & \cdots & d_{i.j,(0,1)}^{n,k} \\ \vdots & \vdots & \ddots & \vdots \\ d_{i.j,(n,k)}^{0,0} & d_{i.j,(n,k)}^{0,1} & \cdots & d_{i.j,(n,k)}^{n,k} \end{bmatrix},\ Z_j = \begin{bmatrix} Z_j^{0,0} \\ Z_j^{0,1} \\ \vdots \\ Z_j^{n,k} \end{bmatrix},\ E_j = \begin{bmatrix} f_j(r_0, s_0) \\ f_j(r_0, s_1) \\ \vdots \\ f_j(r_n, s_k) \end{bmatrix}$$

It is therefore possible to solve system (3.9) by using a standard rule to obtain the unknowns $z_j^{p,q}$ and by using these unknowns in (2.1), we have $B_{n,k}(z_j^{p,q}(r_u, s_v))$, which is the solution of the system of 2DFIEs of the second kind.

IV. CONVERGENCE AND STABILITY ANALYSIS

4.1. Convergence analysis

In this part, we consider some important results about the convergence for 2DFIEs of the first and second kinds.

Theorem 4.1. Consider the system 2DFIEs of the first kind (1.1). Suppose that $H_{ij}(r,s,x,y)$ are continuous on $[a, b]\times[c, d]^2$, and the solution of the equation belong to $(C^{\alpha}\cap L^2)([a, b]\times[c, d])$ for some $\alpha>2$. If the inverse of matrix C, defined in (3.4) exists then

$$\sup_{r_p\in[a,b],\, s_q\in[c,d]}\left|z_j(r_p,s_p)-B_{n,k}(z_j^{p,q})\right|\le \le\sum_{j=1}^{m}\left[\frac{(b-a)^2}{8n}\left\|z''_{j,rr}(r,s)\right\|+\frac{(d-c)^2}{8k}\left\|z''_{j,ss}(r,s)\right\|\right]\left[\beta_{ij}\lambda_{ij}\left\|C^{-1}\right\|(d-c)(b-a)+\frac{1}{m}\right] \tag{4.1}$$

where $r_p=a+\frac{b-a}{n}p,\ p=0,1,\ldots,n,\ s_q=c+\frac{d-c}{k}q,\ q=0,1,\ldots,k$, $z_j(r,s),\ j=1,2,\ldots,m$ is exact solution, $\lambda_{ij}=\sup_{\substack{r,x\in[a,b]\\ s,y\in[c,d]}}\left|\beta_{ij}H_{ij}(r,s,x,y)\right|$, and $B_{n,k}(z_j^{p,q})$ is the proposed method solution.

Proof. It is clear that

$$\begin{aligned}\sup_{r_p\in[a,b],\, s_q\in[c,d]}\left|z_j(r_p,s_p)-B_{n,k}(z_j^{p,q})\right|&\le\\ \le\sup_{r_p\in[a,b],\, s_q\in[c,d]}\left|z_j(r_p,s_p)-B_{n,k}(z_j(r_p,s_q))\right|&+\\ +\sup_{r_p\in[a,b],\, s_q\in[c,d]}\left|B_{n,k}(z_j(r_p,s_q))-B_{n,k}(z_j^{p,q})\right|&\end{aligned} \tag{4.2}$$

From (2.9),

$$\sup_{r_p\in[a,b],\, s_q\in[c,d]}\left|z_j(r_p,s_p)-B_{n,k}(z_j(r_p,s_q))\right|\le\frac{(b-a)^2}{8n}\left\|z''_{j,rr}(r,s)\right\|+\frac{(d-c)^2}{8k}\left\|z''_{j,ss}(r,s)\right\|. \tag{4.3}$$

It remain to find a bound for $\sup\left|B_{n,k}(z_j(r_p,s_q))-B_{n,k}(z_j^{p,q})\right|$.

Now we set $CM=N$ and $CM'=N'$, where $M=[B_{n,k}(z_j^{p,q})],\ N=[f_i(r_p,s_q)]$, $M'=[B_{n,k}(z_j(r_p,s_q)]$, and $N'=[\hat{f}_i(r_p,s_q)]$.

Hence,

$$\begin{aligned}\sup_{r_p\in[a,b],\, s_q\in[c,d]}\left|B_{n,k}(z_j(r_p,s_q))-B_{n,k}(z_j^{p,q})\right|&\le\\ \le\left\|C^{-1}\right\|\sup_{r_p\in[a,b],\, s_q\in[c,d]}\left|f_i(r_p,s_q)-\hat{f}_i(r_p,s_q)\right|&\end{aligned} \tag{4.4}$$

Now we will get a bound for $\sup\left|f_i(r_p,s_q)-\hat{f}_i(r_p,s_q)\right|$. Where

$$f_i(r,s)=\sum_{j=1}^{m}\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)z_j(x,y)dxdy \tag{4.5}$$

$$\hat{f}_i(r,s)=\sum_{j=1}^{m}\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)B_{n,k}(z_j(x,y))dxdy. \tag{4.6}$$

So,

$$f_i(r,s)-\hat{f}_i(r,s)=\sum_{j=1}^{m}\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)[z_j(x,y)-B_{n,k}(z_j(x,y))]dxdy. \tag{4.7}$$

Taking supremum on both sides, we have

$$\begin{aligned}&\sup_{\substack{r,x\in[a,b]\\ s,y\in[c,d]}}\left|f_i(r,s)-\hat{f}_i(r,s)\right|=\\ &=\sup_{\substack{r,x\in[a,b]\\ s,y\in[c,d]}}\left|\sum_{j=1}^{m}\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)[z_j(x,y)-B_{n,k}(z_j(x,y))]dxdy\right|\le\\ &\le\sum_{j=1}^{m}\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)\sup_{\substack{r,x\in[a,b]\\ s,y\in[c,d]}}\left|z_j(x,y)-B_{n,k}(z_j(x,y))\right|dxdy=\\ &=\sum_{j=1}^{m}\beta_{ij}\lambda_{ij}\left(\frac{(b-a)^2}{8n}\left\|z''_{j,rr}(r,s)\right\|+\frac{(d-c)^2}{8k}\left\|z''_{j,ss}(r,s)\right\|\right)(d-c)(b-a)\end{aligned} \tag{4.8}$$

, where $\lambda_{ij}=\sup_{\substack{r,x\in[a,b]\\ s,y\in[c,d]}}\left|\beta_{ij}H_{ij}(r,s,x,y)\right|$.

Putting above result in inequality (4.4), we get for $p=0,1,\ldots,n,\ q=0,1,\ldots,k$

$$\begin{aligned}&\sup_{r_p\in[a,b],\ s_q\in[c,d]}\left|B_{n,k}(z_j(r_p,s_q))-B_{n,k}(z_j^{p,q})\right|\le\\ &\le\left\|C^{-1}\right\|\sum_{j=1}^{m}\beta_{ij}\lambda_{ij}\left[\frac{(b-a)^2}{8n}\left\|z''_{j,rr}(r,s)\right\|+\frac{(d-c)^2}{8k}\left\|z''_{j,ss}(r,s)\right\|\right](d-c)(b-a)\end{aligned} \tag{4.9}$$

Thus, from (4.2), (4.3) and (4.9)

$$\begin{aligned}&\sup_{r_p\in[a,b],\ s_q\in[c,d]}\left|z_j(r_p,s_p)-B_{n,k}(z_j^{p,q})\right|\le\\ &\le\sum_{j=1}^{m}\left[\frac{(b-a)^2}{8n}\left\|z''_{j,rr}(r,s)\right\|+\frac{(d-c)^2}{8k}\left\|z''_{j,ss}(r,s)\right\|\right]\left[\beta_{ij}\lambda_{ij}\left\|C^{-1}\right\|(d-c)(b-a)+\frac{1}{m}\right]\end{aligned} \tag{4.10}$$

The proof is completed. □

That the error bound contains the $\left\|C^{-1}\right\|$ is a disadvantage of this theorem. Hence we find a bound for $\left\|C^{-1}\right\|$ in the following theorem.

Theorem 4.2. In theorem 4.1, assume that $\left\|C-I\right\|=\xi_1<1$, where $\left\|\cdot\right\|$ is the maximum norm of rows and *I* is a identity matrix of order $m(n+1)(k+1)$. Then

$$\left\|C^{-1}\right\| \le \frac{1}{1-\xi_1}.$$

The condition number is

$$Cond(C) \le \frac{\sum_{j=1}^{m}\delta_j}{1-\xi_1},$$

where $\delta_j = \max_{u,v}\left|\beta_{ij}\int_c^d\int_a^b H_{ij}(r_u,s_v,x,y)dxdy\right|$.

Proof. Firstly we determine a bound for $\|C\|$. That is,

$$\|C\| = \max_{u,v}\sum_{j=1}^{m}\sum_{p=0}^{n}\sum_{q=0}^{k}\left|\frac{\binom{n}{p}\binom{k}{q}}{(b-a)^n(d-c)^k}\beta_{ij}\int_c^d\int_a^b H_{ij}(r_u,s_v,x,y)A_{pq}(x,y)dxdy\right| \tag{4.11}$$

, where $A_{pq}(x,y)=(x-a)^p(b-x)^{n-p}(y-c)^q(d-y)^{k-q}$. Since

$$\sum_{p=0}^{n}\sum_{q=0}^{k}\frac{\binom{n}{p}\binom{k}{q}}{(b-a)^n(d-c)^k}A_{pq}(x,y)=1,$$

$$\|C\| = \max_{u,v}\sum_{j=1}^{m}\left|\beta_{ij}\int_c^d\int_a^b H_{ij}(r_u,s_v,x,y)dxdy\right|. \tag{4.12}$$

Hence $\|C\| \le \sum_{j=1}^{m}\delta_j$, where $\delta_j = \max_{u,v}\left|\beta_{ij}\int_c^d\int_a^b H_{ij}(r_u,s_v,x,y)dxdy\right|$.

To set a bound for $\left\|C^{-1}\right\|$, if $L=C-I$, then

$$\left\|C^{-1}\right\| = \left\|(I+L)^{-1}\right\| \le \frac{1}{1-\|L\|} = \frac{1}{1-\xi_1}. \tag{4.13}$$

Thus

$$cond(C) = \|C\|\cdot\left\|C^{-1}\right\| \le \frac{\sum_{j=1}^{m}\delta_{ij}(d-c)(b-a)}{1-\xi_1}$$

The proof is completed. □

Theorem 4.3. Consider the system 2DFIEs of the second kind (1.2). Assume that $H_{ij}(r,s,x,y)$ are continuous on $[a, b]\times[c,$

$d]^2$, and the solution of the equation belong to $(C^{\alpha}\cap L^2)([a,b]\times[c,d])$ for some $\alpha>2$. If the inverse of matrix D, defined in (3.9) exists then

$$\sup_{r_p\in[a,b],\, s_q\in[c,d]}\left|z_j(r_p,s_p)-B_{n,k}(z_j^{p,q})\right| \le \sum_{j=1}^{m}\left[\frac{(b-a)^2}{8n}\left\|z''_{j,rr}(r,s)\right\|+\frac{(d-c)^2}{8k}\left\|z''_{j,ss}(r,s)\right\|\right]\cdot\left[\left\|D^{-1}\right\|\left(u_{ij}(r,s)+\lambda_{ij}(d-c)(b-q)\right)+\frac{1}{m}\right] \quad (4.14)$$

where $r_p=a+\frac{b-a}{n}p,\ p=0,1,\ldots,n,\ s_q=c+\frac{d-c}{k}q,\ q=0,1,\ldots,k$, $z_j(r,s),\ j=1,2,\ldots,m$ is exact solution, $\lambda_{ij}=\sup_{\substack{r,x\in[a,b]\\ s,y\in[c,d]}}\left|\beta_{ij}H_{ij}(r,s,x,y)\right|$, and $B_{n,k}(z_j^{p,q})$ is the proposed method solution.

Proof. Let

$$\begin{aligned}&\sup_{r_p\in[a,b],\, s_q\in[c,d]}\left|z_j(r_p,s_p)-B_{n,k}(z_j^{p,q})\right| \le \\ &\le \sup_{r_p\in[a,b],\, s_q\in[c,d]}\left|z_j(r_p,s_p)-B_{n,k}(z_j(r_p,s_q))\right| + \sup_{r_p\in[a,b],\, s_q\in[c,d]}\left|B_{n,k}(z_j(r_p,s_q))-B_{n,k}(z_j^{p,q})\right|\end{aligned} \quad (4.15)$$

By (2.9),

$$\sup_{r_p\in[a,b],\, s_q\in[c,d]}\left|z_j(r_p,s_p)-B_{n,k}(z_j(r_p,s_q))\right| \le \frac{(b-a)^2}{8n}\left\|z''_{j,rr}(r,s)\right\|+\frac{(d-c)^2}{8k}\left\|z''_{j,ss}(r,s)\right\|. \quad (4.16)$$

Now, determine a bound for $\sup\left|B_{n,k}(z_j(r_p,s_q))-B_{n,k}(z_j^{p,q})\right|$.

Now we set $DQ=P$ and $DQ'=P'$, where $Q=[B_{n,k}(z_j^{p,q})],\ P=[f_i(r_p,s_q)]$, $Q'=[B_{n,k}(z_j(r_p,s_q))]$, and

$P'=[\hat{f}_i(r_p,s_q)]$.

Hence,

$$\begin{aligned}&\sup_{r_p\in[a,b],\, s_q\in[c,d]}\left|B_{n,k}(z_j(r_p,s_q))-B_{n,k}(z_j^{p,q})\right| \le \\ &\le \left\|D^{-1}\right\|\sup_{r_p\in[a,b],\, s_q\in[c,d]}\left|f_i(r_p,s_q)-\hat{f}_i(r_p,s_q)\right|\end{aligned} \quad (4.17)$$

Now we will get a bound for $\sup\left|f_i(r_p,s_q)-\hat{f}_i(r_p,s_q)\right|$. Where

$$f_i(r,s)=\sum_{j=1}^{m}\left[u_{ij}(r,s)z_j(r,s)-\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)z_j(x,y)dxdy\right] \quad (4.18)$$

$$\hat{f}_i(r,s)=\sum_{j=1}^{m}\left[u_{ij}(r,s)B_{n,k}(z_j(r,s))-\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)B_{n,k}(z_j(x,y))dxdy\right] \quad (4.19)$$

Thus,

$$f_i(r,s)-\hat{f}_i(r,s)=\sum_{j=1}^{m}\Big[u_{ij}(r,s)(z_j(r,s)-B_{n,k}(z_j(r,s))-\\ -\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)[z_j(x,y)-B_{n,k}(z_j(x,y))]dxdy\Big] \quad (4.20)$$

Taking supremum on both sides, we have

$$\sup\left|f_i(r,s)-\hat{f}_i(r,s)\right|\le\sum_{j=1}^{m}\Big[u_{ij}(r,s)\sup\left|z_j(r,s)-B_{n,k}(z_j(r,s))\right|+\\ +\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)\sup\left|z_j(x,y)-B_{n,k}(z_j(x,y))\right|dxdy\Big]\le\\ \le\sum_{j=1}^{m}\sup\left|z_j(r,s)-B_{n,k}(z_j(r,s))\right|\cdot\Big[u_{ij}(r,s)+\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)dxdy\Big] \quad (4.21)$$

$$\le\sum_{j=1}^{m}\left[\frac{(b-a)^2}{8n}\left\|z''_{j,rr}(r,s)\right\|+\frac{(d-c)^2}{8k}\left\|z''_{j,ss}(r,s)\right\|\right]\cdot\big[u_{ij}(r,s)+\lambda_{ij}(d-c)(b-a)\big]$$

, where $\lambda_{ij}=\sup_{\substack{r,x\in[a,b]\\ s,y\in[c,d]}}\left|\beta_{ij}H_{ij}(r,s,x,y)\right|$.

Putting above result in inequality (4.17), we get for $p=0,1,\dots,n,\ q=0,1,\dots,k$

$$\sup_{r_p\in[a,b],\,s_q\in[c,d]}\left|B_{n,k}(z_j(r_p,s_q))-B_{n,k}(z_j^{p,q})\right|\le$$

$$\le\left\|D^{-1}\right\|\sum_{j=1}^{m}\left[\frac{(b-a)^2}{8n}\left\|z''_{j,rr}(r,s)\right\|+\frac{(d-c)^2}{8k}\left\|z''_{j,ss}(r,s)\right\|\right]\cdot\big[u_{ij}(r,s)+\lambda_{ij}(d-c)(b-a)\big] \quad (4.22)$$

By using (4.15), (4.16) and (4.22)

$$\sup_{r_p\in[a,b],\,s_q\in[c,d]}\left|z_j(r_p,s_p)-B_{n,k}(z_j^{p,q})\right|\le\frac{(b-a)^2}{8n}\left\|z''_{j,rr}(r,s)\right\|+\frac{(d-c)^2}{8k}\left\|z''_{j,ss}(r,s)\right\|+\\ +\left\|D^{-1}\right\|\sum_{j=1}^{m}\left[\frac{(b-a)^2}{8n}\left\|z''_{j,rr}(r,s)\right\|+\frac{(d-c)^2}{8k}\left\|z''_{j,ss}(r,s)\right\|\right]\cdot\big[u_{ij}(r,s)+\lambda_{ij}(d-c)(b-a)\big]=\\ =\sum_{j=1}^{m}\left[\frac{(b-a)^2}{8n}\left\|z''_{j,rr}(r,s)\right\|+\frac{(d-c)^2}{8k}\left\|z''_{j,ss}(r,s)\right\|\right]\cdot\left[\left\|D^{-1}\right\|\big(u_{ij}(r,s)+\lambda_{ij}(d-c)(b-q)\big)+\frac{1}{m}\right] \quad (4.23)$$

The proof is completed. □

That the error bound contains the $\left\|D^{-1}\right\|$ is a disadvantage of this theorem. Hence we find a bound for $\left\|D^{-1}\right\|$ in the following theorem.

Theorem 4.4. In theorem 4.3, assume that $\left\|D-I\right\|=\xi_2<1$, where $\left\|\cdot\right\|$ is the maximum norm of rows and *I* is a identity matrix of order $m(n+1)(k+1)$. Then

$$\left\|D^{-1}\right\| \leq \frac{1}{1-\xi_2}.$$

The condition number is

$$cond(D) \leq \frac{W + \sum_{j=1}^{m} \delta_j}{1-\xi_2}.$$

Proof. Firstly we find a bound for $\|D\|$. That is,

$$\|D\| = \max_{u,v} \sum_{j=1}^{m} \sum_{p=0}^{n} \sum_{q=0}^{k} \frac{\binom{n}{p}\binom{k}{q}}{(b-a)^n (d-c)^k} \Big| u_{ij}(r_u, s_v) A_{pq}(r_u, s_v) - \beta_{ij} \int_c^d \int_a^b H_{ij}(r_u, s_v, x, y) A_{pq}(x, y) dxdy \Big| \tag{4.24}$$

, where $A_{pq}(x, y) = (x-a)^p (b-x)^{n-p} (y-c)^q (d-y)^{k-q}$. Since the sum of Bernstein basis polynomial is 1, it hold

$$\|D\| = \max_{u,v} \sum_{j=1}^{m} \left| u_{ij}(r_u, s_v) - \beta_{ij} \int_c^d \int_a^b H_{ij}(r_u, s_v, x, y) dxdy \right|. \tag{4.25}$$

Hence

$$\|D\| \leq \max_{u,v} \sum_{j=1}^{m} \left| u_{ij}(r_u, s_v) \right| + \max_{u,v} \sum_{j=1}^{m} \left| \beta_{ij} \int_c^d \int_a^b H_{ij}(r_u, s_v, x, y) dxdy \right|. \tag{4.26}$$

This implies

$$\|D\| \leq W + \sum_{j=1}^{m} \delta_j. \tag{4.27}$$

where $W = \max_{u,v} \sum_{j=1}^{m} \left| u_{ij}(r_u, s_v) \right|$, $\delta_j = \max_{u,v} \left| \beta_{ij} \int_c^d \int_a^b H_{ij}(r_u, s_v, x, y) dxdy \right|$.

To set a bound for $\left\|D^{-1}\right\|$, if $R=D-I$, then

$$\left\|D^{-1}\right\| = \left\|(I+R)^{-1}\right\| \leq \frac{1}{1-\|R\|} = \frac{1}{1-\xi_2}. \tag{4.28}$$

Thus

$$cond(D) = \|D\| \cdot \left\|D^{-1}\right\| \leq \frac{W + \sum_{j=1}^{m} \delta_j}{1-\xi_2}$$

The proof is completed. □

4.2. Stability analysis

The problem of stability analysis con be treated generally for the system of FIEs. Here, Hyers-Ulam stability criteria [26] is used for both first and second kinds of 2DFIEs.

Firstly, let consider stability analysis of 2DFIEs of the second kind in the following theorem.

Theorem 4.5. Assume that $B_{n,k}(z_j(r,s)):[a,b]\times[c,d]\to\mathbf{R},\ f_j(r,s)\in L^2([a,b]\times[c,d])$ and $H_{ij}(r,s,x,y)\in L^2([a,b]\times[c,d])^2$. If $B_{n,k}(z_j(r,s))$ satisfies the following condition

$$\left|\sum_{j=1}^{m}u_{ij}(r,s)B_{n,k}(z_j(r,s))-f_i(r,s)-\sum_{j=1}^{m}\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)B_{n,k}(z_j(x,y))dxdy\right|<\varepsilon\,(\varepsilon>0) \qquad (4.29)$$

where $\left|\sum_{j=1}^{m}\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)dxdy\right|<1$, then there exist the solutions $z_j(r,s)$ satisfying (1.1) and for

$n>\dfrac{2M(b-a)^2}{\varepsilon\delta_1^2}$, $k>\dfrac{2M(d-c)^2}{\varepsilon\delta_2^2}$ and $M=\left\|z_j(r,s)\right\|$, we have $\left\|B_{n,k}(z_j(r,s))-z_j(r,s)\right\|<\varepsilon$.

Proof. Define an operator T by :

$$(T_{z_j})(r,s)=f_i(r,s)+u_{ij}(r,s)+\sum_{j=1}^{m}\beta_{ij}\int_c^d\int_a H_{ij}(r,s,x,y)z_j(r,s)dxdy,\quad z_j(r,s)\in L^2([a,b]\times[c,d]),$$

(4.30)

for i=1,2,…,m. Then, by using Hölder inequality, we have

$$\int_c^d\int_a^b\left[\int_c^d\int_a^b H_{ij}(r,s,x,y)z_j(x,y)dxdy\right]^2 drds\le$$

$$\le\int_c^d\int_a^b\left[\int_c^d\int_a^b[H_{ij}(r,s,x,y)]^2dxdy\cdot\int_c^d\int_a^b[z_j(x,y)]^2dxdy\right]drds\le$$

$$\le\int_c^d\int_a^b[z_j(x,y)]^2dxdy\cdot\int_c^d\int_a^b\int_c^d\int_a^b[H_{ij}(r,s,x,y)]^2dxdydrds<\infty$$

since $Tz_j(r,s)\in L^2([a,b]\times[c,d])$ and T is a self mapping of $L^2([a,b]\times[c,d])$. Hence, the solution of equation (4.30) is the fixed point of mapping T.

Also, $d(T_{u_l},T_{u_m})=\left(\int_c^d\int_a^b\left|Tu_l(r,s)-Tu_m(r,s)\right|^2drds\right)^{\frac{1}{2}}=$

$$=\sum_{j=1}^{m}\left(\int_c^d\int_a^b\left|\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)\{u_l(x,y)-u_m(x,y)\}dxdy\right|^2 drds\right)^{\frac{1}{2}}\le$$

$$\le\sum_{j=1}^{m}|\beta_{ij}|\left(\int_c^d\int_a^b\left\{\int_c^d\int_a^b(H_{ij}(r,s,x,y))^2 dxdy\cdot\int_c^d\int_a^b|u_l(x,y)-u_m(x,y)|^2 dxdy\right\}drds\right)^{\frac{1}{2}}=$$

$$=\sum_{j=1}^{m}|\beta_{ij}|\left(\int_c^d\int_a^b\int_c^d\int_a^b(H_{ij}(r,s,x,y))^2 dxdydrds\right)^{\frac{1}{2}}d(u_l,u_m).$$

And we note that

$$\sum_{j=1}^{m}|\beta_{ij}|\left(\int_c^d\int_a^b\int_c^d\int_a^b(H_{ij}(r,s,x,y))^2 dxdydrds\right)^{\frac{1}{2}}<1.$$

Thus, T is a contractive operator.

From Definition of contractive operator, the equation (4.30) has a unique solution $z_j(r,s)$ satisfying (1.2). Then from Theorem 2.1, we have, $\|B_{n,k}(z_j(r,s))-z_j(r,s)\|<\varepsilon$ for any $n>\dfrac{2M(b-a)^2}{\varepsilon\delta_1^2}$, $k>\dfrac{2M(d-c)^2}{\varepsilon\delta_2^2}$

This completes the proof. □

Similarly, it holds the result as following to the system of 2DFIEs of the first kind.

Theorem 4.6. Assume that $B_{n,k}(z_j(r,s)):[a,b]\times[c,d]\to\mathbf{R}$, $f_j(r,s)\in L^2([a,b]\times[c,d])$ and $H_{ij}(r,s,x,y)\in L^2([a,b]\times[c,d])^2$. If $B_{n,k}(z_j(r,s))$ satisfies the following condition

$$\left|f_i(r,s)-\sum_{j=1}^{m}\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)B_{n,k}(z_j(x,y))dxdy\right|<\varepsilon(\varepsilon>0)\qquad(4.31)$$

where $\left|\sum_{j=1}^{m}\beta_{ij}\int_c^d\int_a^b H_{ij}(r,s,x,y)dxdy\right|<1$, then there exist the solutions $z_j(r,s)$ satisfying (1.1) and for $n>\dfrac{2M(b-a)^2}{\varepsilon\delta_1^2}$, $k>\dfrac{2M(d-c)^2}{\varepsilon\delta_2^2}$ and $M=\|z_j(r,s)\|$, we have $\|B_{n,k}(z_j(r,s))-z_j(r,s)\|<\varepsilon$.

V. CONCLUSION AND FURTHER WORK

The integral equations have rich physical background in recent years. These equations got great interest across many disciplines and widely used in dynamical system with chaotic behavior and quasi-chaotic dynamical systems. The use of Bernstein polynomials to solve initial value problems, boundary value problems, and integral equations has been recently increased because of the fast convergence and less computational cost. In this research the main theme was to provide such a technique which has fast convergence and less computational cost for the solution of the system of both first and second kind 2DFIEs on arbitrary intervals [a; b] . We found that the proposed method gives excellent approximate solutions even by taking a small value of the degree (*n*, *k*). The proposed technique can be extended for the numerical solution of differential equations arising in engineering models, but some adjustments will be require.

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