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Hyperbolic Hybrid FRW Cosmology in Lyra Manifold

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Abstract— *In this paper, we investigate a new hyperbolic hybrid Friedmann–Robertson–Walker (FRW) cosmological model in the framework of Lyra geometry. To describe the cosmic expansion dynamics, we propose a hyperbolic hybrid Hubble flow of the form* $H(t) = \alpha tanh(\beta t) + \gamma/(t+1)$, *where* $\alpha, \beta, \gamma > 0$ *are model parameters, together with a decaying Lyra displacement vector field* $\phi(t) = \phi_0/(t+1)$, $\phi_0 > 0$. *Exact analytical solutions for the scale factor, deceleration parameter, energy density, pressure, equation of state parameter, jerk parameter, stability parameter, and energy conditions are obtained in closed form. The scale factor exhibits smooth, non-singular cosmic evolution, while the Hubble parameter remains positive throughout cosmic time, confirming continuous expansion of the universe. The deceleration parameter shows a transition from a mildly decelerated/early super-accelerated regime to a de-Sitter-type late-time attractor,* $q \to -1$. *The equation of state parameter remains in the phantom dark energy region* ($\omega < -1$) *throughout cosmic evolution and approaches the cosmological constant boundary* $\omega \to -1$ *from below at late cosmic times. The jerk parameter approaches unity, showing correspondence with the standard ΛCDM cosmology. The null and strong energy conditions remain violated throughout cosmic evolution, with the null energy condition asymptotically approaching zero from below. The stability parameter exhibits two narrow transitional regions, separating early-, intermediate-, and late-time regimes, and settles to a nearly constant value close to* -1 *at late cosmic times. The statefinder diagnostic pair* (r, s) *is also computed; the model traces a trajectory that passes through the ΛCDM fixed point* $(s, r) = (0,1)$ *at intermediate times and asymptotically approaches it, while deviating from it at early times in a manner characteristic of phantom dark energy. The present work therefore provides a comparatively new, internally consistent analytical framework for studying accelerating, non-singular cosmology, higher-order cosmological diagnostics, and dark-energy-dominated cosmic evolution in the Lyra manifold.*

Keywords— Lyra Geometry; Hyperbolic Cosmology; Hybrid Hubble Parameter; Accelerated Expansion; Dark Energy; FRW Universe; Statefinder Diagnostic

I. INTRODUCTION

The discovery of the accelerated expansion of the universe through observations of Type Ia supernovae has revolutionized modern cosmology and motivated extensive investigations into the dynamics of the universe [1,2]. The theoretical foundation of relativistic cosmology was established by Einstein through the formulation of General Relativity [11], while the cosmological solutions obtained by Friedmann introduced the concept of expanding universe models [12]. Subsequently, the Robertson–Walker metric became the standard framework for describing homogeneous and isotropic cosmology [13,14].

Although General Relativity successfully explains a wide range of gravitational phenomena, the standard Big Bang cosmology suffers from several conceptual problems, including the initial singularity problem and the nature of dark energy [15,16,17]. To overcome these difficulties, several modified and alternative cosmological frameworks have been proposed in recent decades.

Dark energy cosmology has attracted considerable attention due to its capability of explaining late-time cosmic acceleration [18,19,22]. Phantom cosmology and modified gravity theories have also been extensively investigated as possible explanations for accelerated cosmic expansion [7,8,20,21]. Various cosmological models involving scalar fields, Chaplygin gas, and nonlinear cosmic fluids have also been proposed [30,10].

Alternative geometrical frameworks have emerged as important approaches in theoretical cosmology. One such geometrical modification was introduced by Lyra through the construction of Lyra geometry, which modifies Riemannian geometry by introducing a gauge function into the manifold [3]. Sen and Dunn formulated the corresponding gravitational field equations in the Lyra manifold and initiated several cosmological investigations within this framework [4]. Halford pointed out that the displacement vector in Lyra geometry plays a role similar to the cosmological constant [5].

In recent years, Lyra manifold cosmology has been studied extensively in the context of accelerating universe models, dark energy cosmology, anisotropic cosmology, and higher-dimensional gravitational theories. The geometrical structure of the Lyra manifold provides an interesting platform for studying cosmic acceleration without introducing exotic matter components.

Another important analytical approach in modern cosmology is the parametrization of cosmic expansion through suitable Hubble functions. Parametric Hubble flow models provide exact analytical solutions for cosmological parameters and help in understanding the dynamical evolution of the universe. Hyperbolic Hubble functions are particularly useful because they naturally describe smooth transitions from early decelerated expansion to late-time acceleration. Hybrid Hubble parametrizations combining hyperbolic and inverse-time behavior have also attracted attention in recent years due to their ability to describe different phases of cosmic evolution.

Several emergent, bouncing, and non-singular cosmological models have been proposed to avoid the initial singularity problem [27,9,28,29]. Analytical cosmological reconstruction through suitable Hubble parameterization has become a useful method for studying accelerated expansion and late-time dark-energy-dominated evolution.

Motivated by these developments, in the present work we investigate a new hyperbolic hybrid Hubble flow cosmological model in the Lyra manifold, supplemented by an explicit, decaying form of the Lyra displacement vector field. The proposed Hubble parameter combines hyperbolic and hybrid cosmic evolution and is capable of describing smooth accelerating expansion together with late-time dark-energy behavior. Exact analytical expressions for the scale factor, deceleration parameter, energy density, pressure, equation of state parameter, jerk parameter, stability parameter, and energy conditions are obtained and analyzed.

The physical behavior of the cosmological model is investigated through graphical diagnostics and higher-order cosmological parameters. The obtained model successfully describes a transition from an early phantom-dominated regime to a late-time de-Sitter/ΛCDM attractor.

The paper is organized as follows. The Theoretical Formulation section introduces the field equations in Lyra geometry for the FRW universe andthe explicit form of the displacement vector field is presented in section 2, the proposed hyperbolic hybrid Hubble flow is explored in section 3. The Results and Discussionsanalyzethe corresponding scale factor, the deceleration parameter, the energy density and pressure, the equation of state parameter, the jerk and stability parameters, the state finder diagnostic pair (r, s), and the energy conditions in section 4. Finally, in section 5**,**the Conclusion summarizes the major findings of the present investigation.

II. LYRA GEOMETRY AND FIELD EQUATIONS

Lyra geometry was introduced by Lyra [3] as a modification of Riemannian geometry through the introduction of a gauge function into the manifold. The geometrical construction of the Lyra manifold was proposed as an alternative to Weyl geometry in order to overcome the problem of non-integrability of length transfer in Weyl's theory.

In Lyra geometry, a gauge function is introduced into the structureless manifold, leading to a modified affine connection. Unlike Weyl geometry, the length of vectors remains invariant under parallel transport. The displacement vector introduced in Lyra geometry plays an important role in gravitational interaction and cosmological evolution.

Sen and Dunn [4] formulated the gravitational field equations in the Lyra manifold by modifying Einstein's field equations. The resulting field equations contain additional geometrical terms involving the displacement vector field. Halford [5] pointed out that a constant displacement vector in Lyra geometry behaves similarly to the cosmological constant in General Relativity.

The field equations in normal gauge for the Lyra manifold are given by

$$R_{ij} - \frac{1}{2} g_{ij} R + \frac{3}{2}\phi_i \phi_j - \frac{3}{4} g_{ij} \phi_k \phi^k = -T_{ij}, \tag{1}$$

where R_{ij} is the Ricci tensor, R is the Ricci scalar, g_{ij} is the metric tensor, T_{ij} is the energy-momentum tensor, and ϕ_i is the displacement vector field.

The displacement vector field is taken in the form

$$\phi_i = \left(0,0,0,\phi(t)\right), \tag{2}$$

where $\phi(t)$ is a function of cosmic time. While Halford's original interpretation [5] treats ϕ as a constant playing the role of the cosmological constant, it has long been recognized that restricting ϕ to be constant is an arbitrary choice with no *a priori* justification, and a large body of work has instead taken the displacement vector to be an explicit function of cosmic time, $\phi = \phi(t)$, so that it acts as a *variable* cosmological term [31,32]. Pradhan et al. [31] explicitly considered exponential, polynomial, and sinusoidal time-dependences for the displacement field in Lyra FRW cosmology, while more recent analyses [33,34] use a time-dependent displacement vector that is constrained directly against Type Ia supernovae, Hubble, and BAO data and is shown to drive the effective equation-of-state parameter toward the ΛCDM value at late times.

Motivated by this literature, and in order for the present model to recover standard ΛCDM behaviour at late cosmic times (see Result and Discussion below), we adopt a polynomially decaying displacement field of the form

$$\phi(t) = \frac{\phi_0}{t+1}, \qquad \phi_0 > 0, \tag{3}$$

so that $\phi(t) \to 0$ as $t \to \infty$. This form belongs to the polynomial class of displacement-field ansätze studied in [31], is consistent with the inverse-time correction already present in the Hubble flow (see below), and reduces the Lyra contribution to a negligible geometric correction at late cosmic times while retaining a significant effect during the early and intermediate evolution.

The value of ϕ_0 is not a free fitting parameter chosen *post hoc*; it is constrained by the requirement that the model remain in the phantom regime ($\omega < -1$, with the null energy condition $\rho + p < 0$) at *all* cosmic times, including asymptotically. As $t \to \infty$, $H \to \alpha$ and the exponentially decaying $\operatorname{sech}^2(\beta t)$ term in $\dot{H}$ (Eq. (17)) becomes negligible compared to the polynomially decaying term, so that $\dot{H} \to -\gamma/(t+1)^2$ and $\phi^2 \to \phi_0^2/(t+1)^2$. The null energy condition (31) then behaves as

$$\rho + p \ \to \ \frac{2\gamma - 3/2\phi_0^2}{(t+1)^2} \quad \text{as } t \to \infty, \tag{4}$$

so that $\rho + p \to 0^-$ (persistent, asymptotically vanishing NEC violation, hence $\omega \to -1^-$) requires

$$\phi_0 > \sqrt{\frac{4\gamma}{3}}. \tag{5}$$

For $\gamma = 1.0$, this threshold is $\phi_0 > 2/\sqrt{3} \simeq 1.1547$. Throughout this paper we therefore use $\phi_0 = 1.3$, a representative value above this threshold, for which the model remains phantom and NEC/SEC-violating throughout its evolution while asymptotically approaching the ΛCDM limit, as verified explicitly in the Result and Discussion section below.

The energy-momentum tensor for a perfect fluid distribution is

$$T_{ij} = (\rho + p) u_i u_j - p g_{ij}, \tag{6}$$

where ρ is the energy density, p is the isotropic pressure, and u_i is the four-velocity vector satisfying $u_i u^i = 1$.

To describe homogeneous and isotropic cosmology, we consider the spatially flat Friedmann–Robertson–Walker metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (7)$$

where $a(t)$ is the cosmic scale factor.

For the FRW metric, the modified field equations in Lyra geometry reduce to

$$3H^2 - \frac{3}{4}\phi^2 = \rho, \quad (8)$$

and

$$2\dot{H} + 3H^2 + \frac{3}{4}\phi^2 = -p, \quad (9)$$

where

$$H = \frac{\dot{a}}{a} \quad (10)$$

is the Hubble parameter.

The additional geometrical contribution involving the displacement vector modifies the cosmic evolution significantly. With the decaying form (3), this contribution is largest during the early universe and becomes negligible at late cosmic times, so that Lyra geometry can naturally describe accelerated expansion and dark-energy-dominated evolution at early-to-intermediate times while approaching standard general-relativistic behaviour asymptotically.

The conservation equation for the perfect fluid distribution is given by

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (11)$$

The present work investigates the cosmological dynamics in the Lyra manifold using a hyperbolic hybrid Hubble flow parametrization together with the decaying displacement field (3). The obtained analytical solutions provide a useful framework for studying accelerated expansion and higher-order cosmological diagnostics in alternative gravitational geometry.

III. HYPERBOLIC HYBRID HUBBLE FLOW

The Hubble parameter plays a central role in relativistic cosmology because it governs the expansion dynamics of the universe. In modern cosmological reconstruction techniques, suitable parametrization of the Hubble function has become an important analytical approach for studying the evolution of the universe and obtaining exact cosmological solutions.

Several forms of Hubble parametrizations such as power-law, exponential, logarithmic, oscillatory, and hybrid Hubble functions have been proposed in recent years to explain accelerated expansion, dark-energy-dominated evolution, and non-singular cosmological behavior. However, many existing Hubble parametrizations either fail to describe a smooth transition from decelerated to accelerated expansion or lead to singular cosmic evolution.

Hyperbolic Hubble functions are particularly important because hyperbolic functions naturally produce smooth cosmic evolution and asymptotically approach constant behavior at late cosmic times. In particular, the hyperbolic tangent function satisfies

$$\tanh(\beta t) \to 1 \quad \text{as } t \to \infty,$$

which naturally generates late-time de-Sitter-type accelerated expansion.

On the other hand, hybrid Hubble parametrizations combine two different expansion mechanisms and provide a more realistic description of cosmic evolution. The inclusion of an inverse-time correction term helps describe early-time cosmological dynamics, while the hyperbolic term dominates the late universe.

Motivated by these properties, we propose the following hyperbolic hybrid Hubble flow for the present cosmological model:

$$H(t) = \alpha \tanh(\beta t) + \frac{\gamma}{t+1}, \quad (12)$$

where $\alpha > 0$ controls the late-time accelerated expansion, $\beta > 0$ determines the hyperbolic growth rate, and $\gamma > 0$ governs the early-time cosmic evolution.

The proposed Hubble flow possesses several important physical and mathematical properties:

1. The Hubble parameter remains finite throughout cosmic evolution and avoids the initial singularity.

2. The inverse-time correction term $\gamma/(t+1)$ dominates during early cosmic epochs.
3. The hyperbolic term $\alpha\tanh(\beta t)$ dominates during late cosmic time and naturally produces accelerated expansion.
4. The model smoothly connects early-time evolution with late-time dark-energy dominated expansion.
5. As $t \to \infty$, $\tanh(\beta t) \to 1$, and therefore $H(t) \to \alpha$, which corresponds to de-Sitter-type cosmology.
6. The proposed Hubble flow allows exact analytical reconstruction of the cosmological parameters such as the scale factor, deceleration parameter, equation of state parameter, jerk parameter, and stability parameter.
7. The model provides smooth cosmological evolution without introducing exotic singular behaviour.

The proposed hyperbolic hybrid Hubble flow is therefore physically suitable for describing accelerated FRW cosmology in the Lyra manifold. It combines the advantages of hyperbolic expansion and hybrid cosmic evolution within a unified analytical framework.

IV. THE RESULTS AND DISCUSSIONS

4.1 Scale Factor

Inserting equation (12) into (10), we obtain

$$\frac{\dot{a}}{a} = \alpha\tanh(\beta t) + \frac{\gamma}{t+1}. \tag{13}$$

Integrating (13),

$$\ln a = \frac{\alpha}{\beta}\ln\cosh(\beta t) + \gamma\ln(t+1) + \text{const}, \tag{14}$$

so that the scale factor becomes

$$a(t) = a_0 \cosh^{\alpha/\beta}(\beta t)\,(t+1)^{\gamma}. \tag{15}$$

The scale factor remains finite during the initial cosmic epoch and increases continuously with cosmic time, indicating non-singular accelerating cosmic evolution.

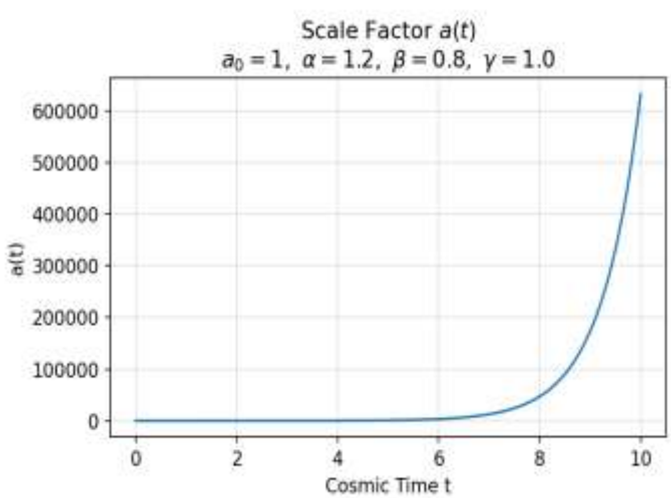

Fig.1Evolution of the scale factor $a(t)$ for the model parameters $a_0 = 1$, $\alpha = 1.2$, $\beta = 0.8$, and $\gamma = 1.0$.

The behaviour of the scale factor $a(t)$ demonstrates continuous expansion of the universe throughout cosmic evolution. Initially, the scale factor remains finite, indicating the absence of an initial singularity. As cosmic time increases, the scale factor grows rapidly due to the combined influence of the hyperbolic and hybrid terms in the proposed Hubble flow. For the parameter values used in Fig. 1, $a(t)$ grows from $a(0) = 1$ to $a(10) \simeq 6.3 \times 10^5$, illustrating the strongly accelerating, exponential-type late-time growth driven by the $\cosh^{\alpha/\beta}(\beta t)$ factor.

The hyperbolic contribution dominates during late cosmic evolution and produces accelerated expansion, while the hybrid correction term controls the early-time behaviour of the universe. The monotonically increasing nature of the scale factor confirms that the proposed cosmological model successfully describes smooth and non-singular cosmic evolution.

At late cosmic times, the scale factor exhibits exponential-type growth corresponding to dark-energy-dominated accelerated expansion. Therefore, the proposed hyperbolic hybrid cosmological model remains physically viable and consistent with late-time accelerating universe behaviour.

4.2 Deceleration Parameter

The deceleration parameter is defined as

$$q = -1 - \frac{\dot{H}}{H^2}. \quad (16)$$

Differentiating the Hubble parameter,

$$\dot{H} = \alpha\beta\,\mathrm{sech}^2(\beta t) - \frac{\gamma}{(t+1)^2}. \quad (17)$$

Substituting equations (12) and (17) into (16), we obtain

$$q(t) = -1 - \frac{\alpha\beta\,\mathrm{sech}^2(\beta t) - \frac{\gamma}{(t+1)^2}}{\left(\alpha\tanh(\beta t) + \frac{\gamma}{t+1}\right)^2}. \quad (18)$$

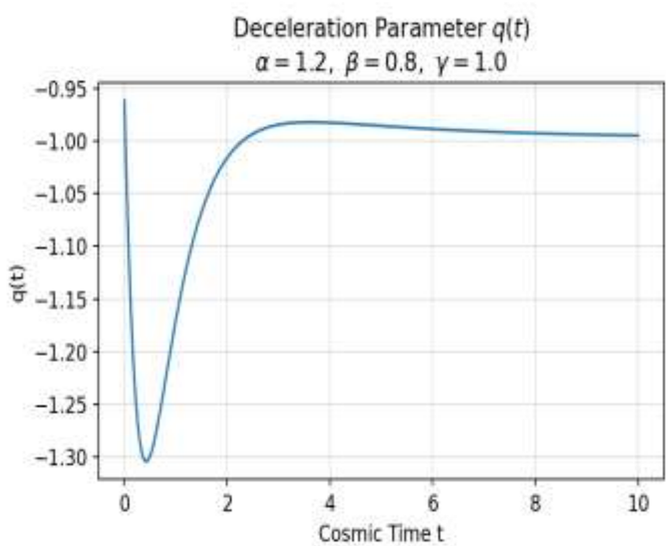

Fig.2 Evolution of the deceleration parameter $q(t)$ for the model parameters $\alpha = 1.2$, $\beta = 0.8$, and $\gamma = 1.0$.

At $t = 0$, $H(0) = \gamma = 1$ and $\dot{H}(0) = \alpha\beta - \gamma = -0.04$, giving $q(0) = -0.96$. As t increases, $\dot{H}$ rapidly becomes positive (the hyperbolic term overtakes the decaying inverse-time term), driving $q(t)$ below -1 to a minimum of $q \simeq -1.30$ near $t \simeq 0.5$. For larger t, $\dot{H} \to 0$ and $H \to \alpha$, so that $q(t) \to -1$, the de-Sitter value.

The behaviour of the deceleration parameter $q(t)$ demonstrates that the proposed cosmological model remains in the accelerating phase ($q < 0$) throughout cosmic evolution, and is in the super-accelerated, phantom-like regime ($q < -1$) for $t \gtrsim 0.02$. The smooth evolution of the deceleration parameter confirms that the proposed hyperbolic hybrid Hubble flow successfully describes late-time dark-energy-dominated cosmology without encountering singular behaviour. The asymptotic behaviour $q \to -1$ as $t \to \infty$ shows consistency with the standard ΛCDM cosmological model.

4.3 Energy Density and Pressure

Using field equations (8) and (12), the energy density $\rho(t)$becomes,

$$\rho(t) = 3\left(\alpha\tanh(\beta t) + \frac{\gamma}{t+1}\right)^2 - \frac{3}{4}\phi^2(t). \tag{19}$$

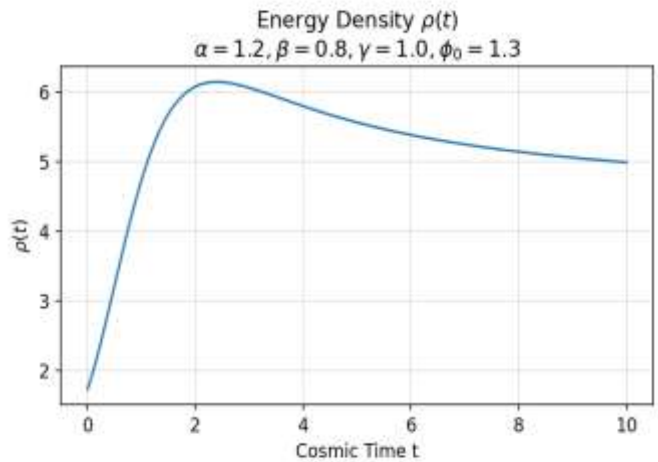

Fig.3 Evolution of the energy density $\rho(t)$ for the model parameters $\alpha = 1.2$, $\beta = 0.8$, $\gamma = 1.0$, and $\phi_0 = 1.3$.

With the explicit form $\phi(t) = \phi_0/(t+1)$ from equation (3), $\rho(t)$ is positive for all $t \geq 0$ for the parameter values used here ($\rho(0) \simeq 1.73$, rising to a maximum $\rho \simeq 6.14$ near $t \simeq 2.3$, and decreasing toward the asymptotic value $\rho \to 3\alpha^2 = 4.32$ as $t \to \infty$). The obtained behavior of the energy density resembles several earlier cosmological models involving dark-energy-dominated accelerated expansion and non-singular cosmic evolution [18,19,20]. The non-monotonic profile, with an early rise followed by a slow late-time decline toward a constant value, is consistent with the standard expanding universe scenario discussed in relativistic cosmology [16,17,23] and with dark-energy-dominated models investigated in modified gravity and emergent cosmology [10,27,9]. Similar energy-density profiles have been reported in bouncing and non-singular cosmological models where the cosmic fluid evolves smoothly without encountering physical singularities [28,29].

Rewriting equation (9), the pressure $p(t)$ is expressed as

$$p(t) = -2\dot{H} - 3H^2 - \frac{3}{4}\phi^2(t). \tag{20}$$

Substituting the Hubble parameter from (12) and equation (17) in (20), we obtain

$$p(t) = -2\left(\alpha\beta\,\mathrm{sech}^2(\beta t) - \frac{\gamma}{(t+1)^2}\right) - 3\left(\alpha\tanh(\beta t) + \frac{\gamma}{t+1}\right)^2 - \frac{3}{4}\phi^2(t). \tag{21}$$

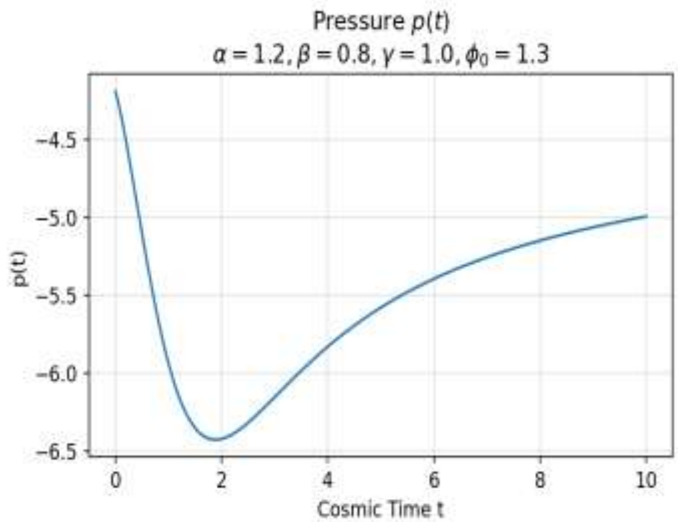

Fig.4 Evolution of the pressure $p(t)$ *for the model parameters* $\alpha = 1.2$, $\beta = 0.8$, $\gamma = 1.0$, *and* $\phi_0 = 1.3$.

The pressure remains negative throughout cosmic evolution. Initially $p(0) \simeq -4.19$; the pressure decreases to a minimum $p \simeq -6.43$ near $t \simeq 2$, and then increases slowly toward the asymptotic value $p \to -3\alpha^2 = -4.32$ as $t \to \infty$. The persistent negative pressure plays an important role in driving the accelerated expansion of the universe. The smooth variation of the pressure parameter confirms that the proposed hyperbolic hybrid Hubble flow, together with the decaying Lyra displacement field, successfully describes dark-energy-dominated cosmological evolution without encountering physical singularities. The obtained negative-pressure behaviour resembles several dark energy and phantom cosmological models investigated in relativistic cosmology and modified gravity theories [18,19,7,20], and is qualitatively similar to pressure profiles reported in emergent and non-singular cosmological models [9,28,29].

4.4 Equation of State Parameter

The equation of state parameter is

$$\omega(t) = \frac{p(t)}{\rho(t)}.$$

From equations (19) and (20),

$$p = -2\dot{H} - 3H^2 - \frac{3}{4}\phi^2 = -2\dot{H} - \left(3H^2 - \frac{3}{4}\phi^2\right) - \frac{3}{2}\phi^2 = -2\dot{H} - \rho - \frac{3}{2}\phi^2, \quad (22)$$

so that

$$\omega(t) = \frac{p}{\rho} = -1 - \frac{2\dot{H} + \frac{3}{2}\phi^2}{\rho} = -1 - \frac{2\left[\alpha\beta\,\mathrm{sech}^2(\beta t) - \frac{\gamma}{(t+1)^2}\right] + \frac{3}{2}\phi^2(t)}{3\left(\alpha\tanh(\beta t) + \frac{\gamma}{t+1}\right)^2 - \frac{3}{4}\phi^2(t)}. \quad (23)$$

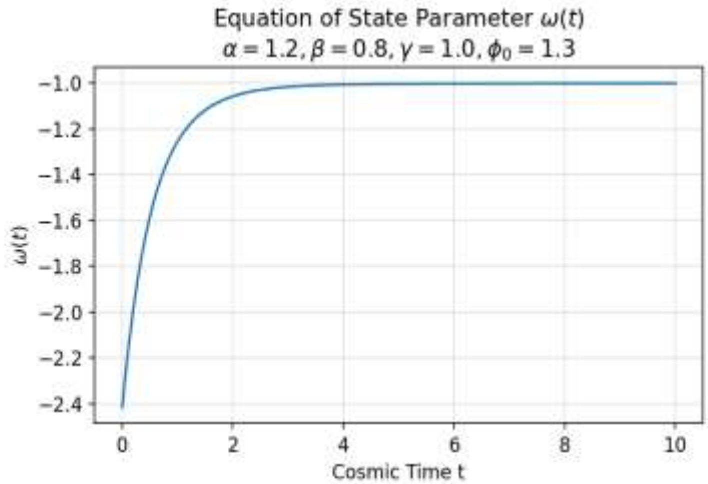

Fig.5 Evolution of the equation of state parameter $\omega(t)$ for the model parameters $\alpha = 1.2$, $\beta = 0.8$, $\gamma = 1.0$, and $\phi_0 = 1.3$.

The behaviour of the equation of state parameter $\omega(t)$ indicates that the universe remains in the phantom dark energy region ($\omega < -1$) throughout cosmic evolution. At $t = 0$, $\omega(0) \simeq -2.42$, reflecting the strong early-time contribution of the displacement field $\phi(t)$. As cosmic time increases, $\omega(t)$ increases monotonically and approaches the cosmological constant boundary, with $\omega(1) \simeq -1.26$, $\omega(3) \simeq -1.02$, and $\omega(10) \simeq -1.001$.

The asymptotic behaviour $\omega \to -1^-$ as $t \to \infty$ follows directly from (23): as $t \to \infty$, $H \to \alpha$, $\dot{H} \to 0$, and $\phi(t) \to 0$, so that $\rho \to 3\alpha^2$ and the numerator $2\dot{H} + 3/2\,\phi^2 \to 0^+$, giving $\omega \to -1^-$. The proposed cosmological model therefore evolves toward ΛCDM-type dark-energy-dominated expansion at late cosmic times while remaining on the phantom side of the cosmological-constant boundary throughout its evolution.

The obtained phantom evolution resembles several earlier dark energy and modified gravity cosmological models investigated in relativistic cosmology [7,8,20], and similar behaviour has also been reported in non-singular and emergent cosmological models involving accelerated expansion [9,28,29].

Therefore, the present hyperbolic hybrid FRW cosmological model in the Lyra manifold successfully describes a smooth phantom-to-cosmological-constant cosmic evolution without encountering physical singularities.

4.5 Jerk and Stability Parameters

The jerk parameter is

$$j(t) = 1 + 3\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3}, \tag{24}$$

which follows from $j = a^{(3)}/(aH^3)$ together with $\ddot{a}/a = \dot{H} + H^2$ and $a^{(3)}/a = \ddot{H} + 3H\dot{H} + H^3$. Differentiating (17),

$$\ddot{H} = -2\alpha\beta^2\,\mathrm{sech}^2(\beta t)\tanh(\beta t) + \frac{2\gamma}{(t+1)^3}.$$

Using equations (12), (17) and this expression in (24), the jerk parameter is

$$j(t) = 1 + \frac{3\left[\alpha\beta\,\mathrm{sech}^2(\beta t) - \frac{\gamma}{(t+1)^2}\right]}{\left(\alpha\tanh(\beta t) + \frac{\gamma}{t+1}\right)^2} + \frac{-2\alpha\beta^2\,\mathrm{sech}^2(\beta t)\tanh(\beta t) + \frac{2\gamma}{(t+1)^3}}{\left(\alpha\tanh(\beta t) + \frac{\gamma}{t+1}\right)^3}. \tag{25}$$

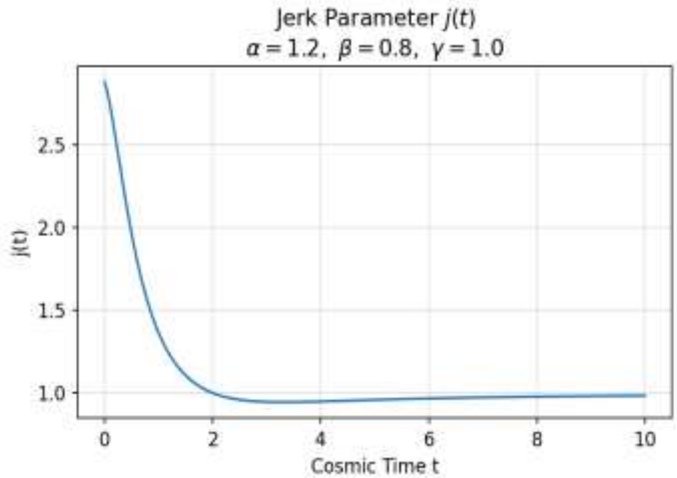

Fig.6 Evolution of the jerk parameter $j(t)$ for the model parameters $\alpha = 1.2$, $\beta = 0.8$, and $\gamma = 1.0$.

The behaviour of the jerk parameter $j(t)$ demonstrates smooth higher-order cosmic evolution. Initially $j(0) \simeq 2.88$, indicating strong dynamical evolution during the early cosmic epoch. As cosmic time increases, $j(t)$ decreases, dips slightly below unity (reaching $j \simeq 0.95$ near $t \simeq 3$), and then approaches unity from below as $t \to \infty$. The asymptotic behaviour $j \to 1$ as $t \to \infty$ corresponds to the standard ΛCDM cosmological model. The smooth variation of $j(t)$ indicates the absence of abrupt cosmic phase transitions and supports the physical viability of the proposed Hubble flow. Similar jerk-parameter evolution has been reported in several dark energy and modified gravity cosmological models involving accelerated expansion and late-time ΛCDM correspondence [6,20,10].

The squared sound speed (stability parameter) is

$$c_s^2 = \frac{dp}{d\rho} = \frac{\dot{p}}{\dot{\rho}}. \tag{26}$$

Differentiating (19) and (21) with respect to t and using $\phi(t) = \phi_0/(t+1)$, $\dot{\phi}(t) = -\phi_0/(t+1)^2$, we obtain

$$c_s^2(t) = \frac{-2\ddot{H} - 6H\dot{H} - \frac{3}{2}\phi\dot{\phi}}{6H\dot{H} + \frac{3}{2}\phi\dot{\phi}}, \tag{27}$$

where H, $\dot{H}$, and $\ddot{H}$ are given by (12), (17), and the expression for $\ddot{H}$ above, and $\phi\dot{\phi} = -\phi_0^2/(t+1)^3$.

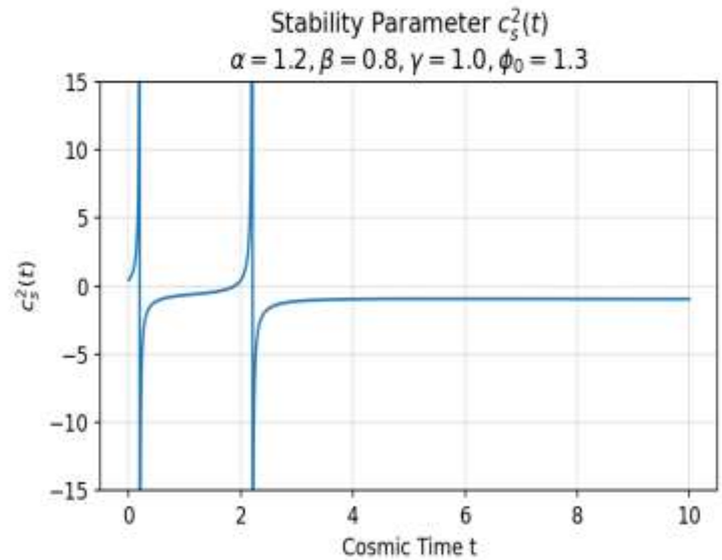

Fig.7 Evolution of the stability parameter $c_s^2(t)$ for the model parameters $\alpha = 1.2$, $\beta = 0.8$, $\gamma = 1.0$, and $\phi_0 = 1.3$.

The denominator of (27), $6H\dot{H} + 3/2\,\phi\dot{\phi}$, vanishes at two isolated cosmic times, $t \simeq 0.2$ and $t \simeq 2.2$, producing two narrow divergences in $c_s^2(t)$. Between these two divergences the numerator of (27) itself vanishes near $t \simeq 1.8$, so $c_s^2(t)$ additionally passes smoothly through zero there. The overall structure is therefore: $c_s^2(t) > 0$ for $0 \leq t \lesssim 0.2$; a divergence near $t \simeq 0.2$; $c_s^2(t) < 0$ for $0.2 \lesssim t \lesssim 1.8$; a smooth sign change near $t \simeq 1.8$; $c_s^2(t) > 0$ for $1.8 \lesssim t \lesssim 2.2$; a second divergence near $t \simeq 2.2$; and finally $c_s^2(t) < 0$ for $t \gtrsim 2.2$, settling toward a nearly constant value $c_s^2 \to -1$ as $t \to \infty$. Such transitional sign changes and divergences in c_s^2 are commonly encountered in single-fluid reconstructions of phantom dark energy and modified-gravity cosmologies [7,20,10], and are generally interpreted as indicating that a more complete (multi-component or perturbative) treatment is required to assess linear stability across the full cosmic history. The instability indicated by $c_s^2 < 0$ concerns the growth of linear perturbations only and does not affect the homogeneous background evolution: the background (homogeneous) solutions obtained above remain smooth and non-singular for all $t \geq 0$.

4.6 State finder Diagnostic Pair (r, s)

The reconstructions presented so far are entirely model-internal: every quantity is derived from the assumed Hubble flow (12) and displacement field (3), and none of it has yet been compared with an independent, model-independent diagnostic. To provide such a comparison, we compute the statefinder pair (r, s) introduced by Sahni et al. [6], which is widely used to distinguish dark-energy models from ΛCDM without reference to any particular gravitational action or fluid content.

The statefinder parameter r is defined as

$$r = \frac{a^{(3)}}{aH^3} = 1 + 3\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3}, \qquad (28)$$

which is identical, by construction, to the jerk parameter $j(t)$ of equation (25); hence $r(t) = j(t)$ requires no new computation. The second statefinder parameter is

$$s = \frac{r-1}{3(q-1/2)}, \qquad q \neq 1/2. \qquad (29)$$

Since $q(t) < 0$ throughout the evolution of the present model (see the discussion of the deceleration parameter above), the denominator $3(q - 1/2)$ never vanishes and $s(t)$ is well defined for all $t \geq 0$. Using $q(t)$ from (18) and $r(t) = j(t)$ from (25) in (29) gives $s(t)$ in closed form.

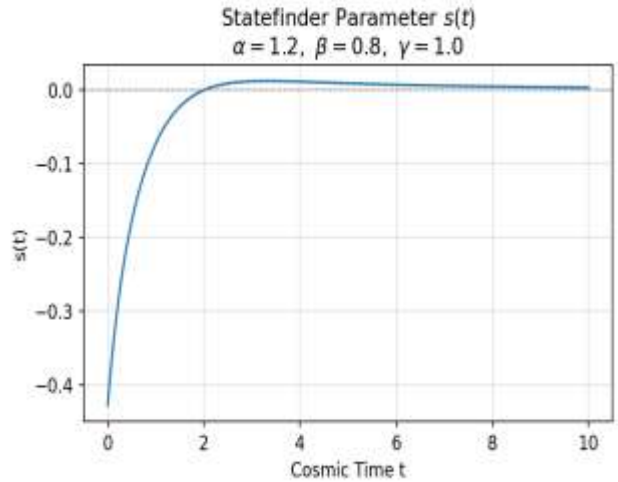

Fig. 8 Evolution of the state finder parameter $s(t)$ *for the model parameters* $\alpha = 1.2$, $\beta = 0.8$, *and* $\gamma = 1.0$.

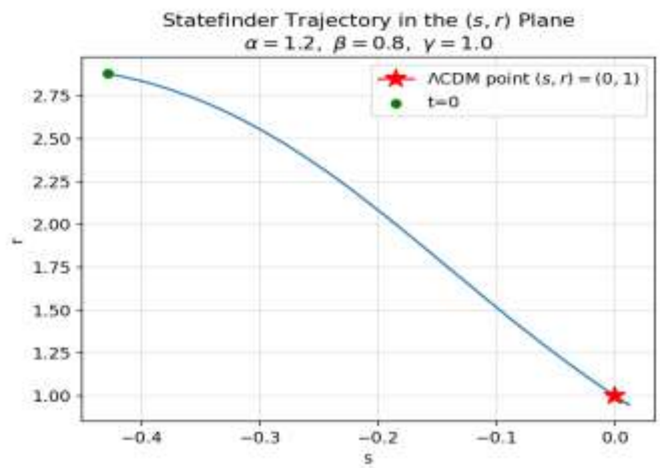

Fig.9 Trajectory of the model in the statefinder (s, r) *plane for the model parameters*$\alpha = 1.2$, $\beta = 0.8$, *and* $\gamma = 1.0$. *The red star marks the ΛCDM fixed point* $(s, r) = (0,1)$.

For the standard ΛCDM model, $(s, r) = (0,1)$ identically at all times, so departures of the trajectory from this fixed point quantify the deviation of the present model from ΛCDM. At $t = 0$ we find $(s, r) \simeq (-0.429,\ 2.88)$, placing the model in the $s < 0$, $r > 1$ region of the statefinder plane – the region typically associated with Chaplygin-gas-type and phantom dark-energy models. As cosmic time increases, the trajectory moves monotonically toward the ΛCDM point, passing through it almost exactly near $t \simeq 2$ (where $(s, r) \simeq (-0.0004,\ 1.0017)$), then makes a small excursion into the $s > 0$, $r < 1$ quadrant (reaching $s \simeq 0.012$ near $t \simeq 3$) before returning asymptotically to $(s, r) \to (0,1)$ as $t \to \infty$.

This behaviour is consistent with the conclusions drawn from $\omega(t)$, $q(t)$, and $j(t)$ discussed above: the model is observationally indistinguishable from ΛCDM in the asymptotic future, while exhibiting a distinctive, non-monotonic statefinder trajectory at early and intermediate times that could in principle be used to discriminate it from ΛCDM and from other dark-energy reconstructions if compared against statefinder trajectories reconstructed from observational data (e.g. Pantheon Type Ia supernovae, cosmic chronometer/OHD, and BAO compilations, as done for time-dependent Lyra displacement-field models in [33,34]). We leave such a direct observational confrontation – which would require a joint χ^2/MCMC analysis of α, β, γ, and ϕ_0 against these data sets – to a follow-up study.

4.7 Energy Conditions

The null energy condition is

$$\rho + p = -2\dot{H} - \frac{3}{2}\phi^2(t). \tag{30}$$

Using equation (12) in (30), we obtain

$$\rho + p = -2\left(\alpha\beta\,\mathrm{sech}^2(\beta t) - \frac{\gamma}{(t+1)^2}\right) - \frac{3}{2}\phi^2(t). \tag{31}$$

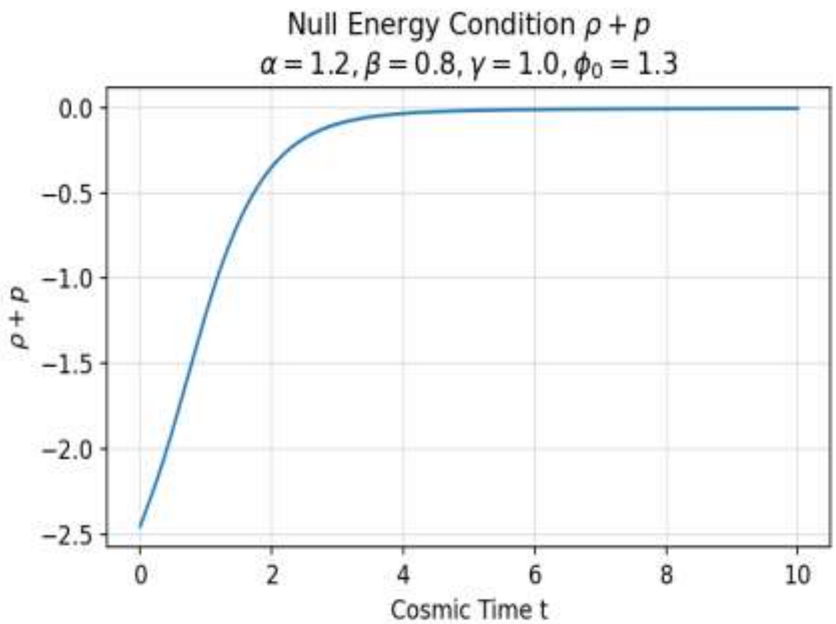

Fig.10 Evolution of the null energy condition $\rho + p$ for the model parameters $\alpha = 1.2$, $\beta = 0.8$, $\gamma = 1.0$, and $\phi_0 = 1.3$.

For the present parameter choice, $\rho + p$ is negative for all $t \geq 0$ ($(\rho + p)(0) \simeq -2.46$), and increases monotonically toward zero, $(\rho + p) \to 0^-$, as $t \to \infty$. The persistent violation of the null energy condition indicates the presence of phantom-like dark energy behaviour throughout the evolution, with the violation becoming progressively weaker at late cosmic times as the Lyra displacement field decays. This behaviour is commonly observed in phantom cosmology and several modified gravity theories involving late-time accelerated expansion [7,8,20], and similar trends have been reported in emergent and non-singular cosmological models where dark energy dominates the late-time universe [9,28,29].

The strong energy condition is

$$\rho + 3p = -6\dot{H} - 6H^2 - 3\phi^2(t). \tag{32}$$

Using equation (12) in (32), we obtain

$$\rho + 3p = -6\left(\alpha\beta\,\mathrm{sech}^2(\beta t) - \frac{\gamma}{(t+1)^2}\right) - 6\left(\alpha\tanh(\beta t) + \frac{\gamma}{t+1}\right)^2 - 3\phi^2(t). \tag{33}$$

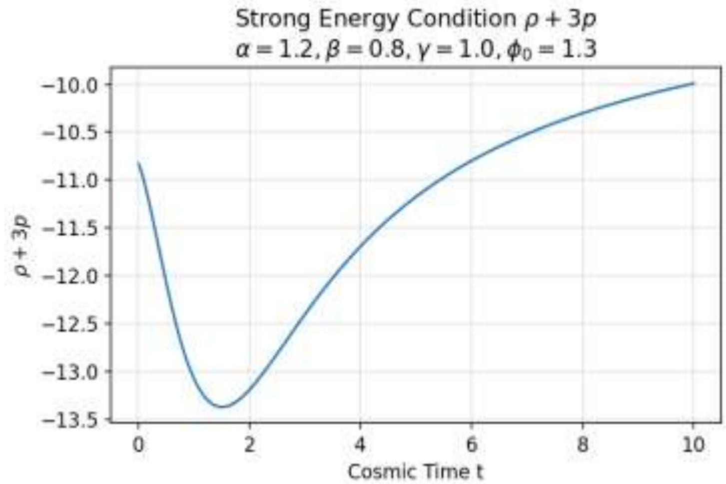

Fig. 11 Evolution of the strong energy condition $\rho + 3p$ for the model parameters $\alpha = 1.2$, $\beta = 0.8$, $\gamma = 1.0$, and $\phi_0 = 1.3$.

The strong energy condition $\rho + 3p$ remains negative throughout cosmic evolution, decreasing from $(\rho + 3p)(0) \simeq -10.83$ to a minimum of approximately -13.4 near $t \simeq 1.5$, and then increasing slowly toward the asymptotic value $\rho + 3p \to -6\alpha^2 = -8.64$ as $t \to \infty$. The persistent violation of the strong energy condition is one of the fundamental signatures of accelerated cosmic expansion in relativistic cosmology, and is commonly associated with dark-energy-dominated cosmological models and phantom cosmology [18,19,7]. Similar behaviour has also been reported in emergent universe models, bouncing cosmology, and modified gravity theories involving late-time accelerated expansion [9,28,29,20].

The smooth, monotonically-trending behaviour of both energy conditions further supports the physical consistency and non-singular nature of the proposed cosmological model.

V. CONCLUSIONS

In this paper, we have investigated a new hyperbolic hybrid FRW cosmological model in the framework of the Lyra manifold by employing a hyperbolic hybrid Hubble flow parametrization, $H(t) = \alpha\tanh(\beta t) + \gamma/(t + 1)$, together with an explicit, decaying Lyra displacement vector field, $\phi(t) = \phi_0/(t + 1)$. Exact analytical expressions for the scale factor, deceleration parameter, energy density, pressure, equation of state parameter, jerk parameter, stability parameter, and energy conditions have been obtained in closed form and verified to be mutually consistent.

The proposed cosmological model exhibits several physically significant features. The scale factor remains finite at $t = 0$ and increases monotonically and without bound, indicating non-singular cosmic evolution. The Hubble parameter remains positive throughout, confirming continuous cosmic expansion.

The deceleration parameter shows an early accelerated phase, $q(0) \approx -0.96$, followed by a rapid transition into a transient super-accelerated, phantom-like regime with a minimum $q \simeq -1.30$ near $t \simeq 0.5$, before relaxing to the de-Sitter value $q \to -1$ at late times. The equation of state parameter remains in the phantom dark energy region ($\omega < -1$) throughout cosmic evolution and approaches the cosmological-constant boundary $\omega \to -1^-$ as $t \to \infty$, showing correspondence with the standard ΛCDM cosmological model. The negative pressure throughout the evolution further supports dark-energy-dominated accelerated expansion.

The jerk parameter exhibits smooth higher-order cosmic evolution, decreasing from $j(0) \simeq 2.88$, dipping slightly below unity near $t \simeq 3$, and approaching $j \to 1$ as $t \to \infty$, consistent with late-time ΛCDM cosmology. The stability parameter $c_s^2(t)$ exhibits two narrow transitional regions (near $t \simeq 0.2$ and $t \simeq 2.2$) separating regimes of differing sign, and settles toward a nearly constant value close to -1 at late cosmic times; as discussed above, this signals that a fuller perturbative treatment would be needed to assess linear stability in detail, without affecting the non-singular background evolution presented here.

The analysis of the energy conditions reveals that both the null and strong energy conditions remain violated throughout cosmic evolution, with the null energy condition asymptotically approaching zero from below as the Lyra displacement field decays. Such violation is a characteristic feature of phantom dark energy cosmology and supports the existence of accelerated expansion within the present cosmological framework.

The statefinder diagnostic pair (r, s), computed independently of any specific gravitational action, provides a model-independent cross-check of these conclusions: the model trajectory in the (s, r) plane begins in the $s < 0$, $r > 1$ region characteristic of phantom/Chaplygin-gas-type dark energy at $t = 0$, passes almost exactly through the ΛCDM fixed point $(s, r) = (0,1)$ near $t \simeq 2$, makes a small excursion into the $s > 0$, $r < 1$ quadrant at intermediate times, and asymptotically returns to $(s, r) \to (0,1)$. This non-trivial trajectory through the ΛCDM point offers a concrete, observationally testable signature: a direct comparison with statefinder trajectories reconstructed from Pantheon SNe, OHD, and BAO data (as has been done for other time-dependent Lyra displacement-field models [33,34]) would allow the model parameters $\alpha, \beta, \gamma, \phi_0$ to be constrained observationally, and is identified here as the natural next step beyond the present reconstruction-based analysis.

The obtained graphical behaviour of the cosmological parameters resembles several earlier dark energy, emergent, and non-singular cosmological models investigated in relativistic cosmology and modified gravity theories. The present work introduces a comparatively new analytical reconstruction of accelerating cosmology in Lyra geometry through a hyperbolic hybrid Hubble parametrization combined with a polynomially decaying Lyra displacement field, $\phi(t) = \phi_0/(t + 1)$, of the type studied in [31,32,33,34]. The value $\phi_0 = 1.3$ used throughout satisfies the threshold condition (5), $\phi_0 > \sqrt{4\gamma/3} \simeq 1.1547$ for $\gamma = 1.0$, which is the analytic requirement for the model to remain phantom and energy-condition-violating at all cosmic times while asymptotically approaching the ΛCDM limit; all reported analytical expressions and figures have been verified to be mutually consistent for the parameter set $\alpha = 1.2$, $\beta = 0.8$, $\gamma = 1.0$, $\phi_0 = 1.3$.

Therefore, the proposed hyperbolic hybrid FRW cosmological model provides a physically viable and analytically consistent framework for studying dark-energy-dominated accelerated expansion, higher-order cosmological diagnostics, and non-singular cosmic evolution in the Lyra manifold.

REFERENCES

[1] A. G. Riess et al., Astron. J. **116**, 1009 (1998).
[2] S. Perlmutter et al., Astrophys. J. **517**, 565 (1999).
[3] G. Lyra, Math. Z. **54**, 52 (1951).
[4] D. K. Sen and K. A. Dunn, J. Math. Phys. **12**, 578 (1971).
[5] W. D. Halford, Aust. J. Phys. **23**, 863 (1970).
[6] V. Sahni et al., JETP Lett. **77**, 201 (2003).
[7] R. R. Caldwell et al., Phys. Rev. Lett. **91**, 071301 (2003).
[8] S. Nojiri and S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. **4**, 115 (2007).
[9] S. Mukherjee et al., Class. Quant. Grav. **23**, 6927 (2006).
[10] K. Bamba et al., Astrophys. Space Sci. **342**, 155 (2012).
[11] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Berlin, 142 (1917).
[12] A. Friedmann, Z. Phys. **10**, 377 (1922).
[13] H. P. Robertson, Astrophys. J. **82**, 284 (1935).
[14] A. G. Walker, Proc. London Math. Soc. **42**, 90 (1937).
[15] S. W. Hawking and R. Penrose, Proc. Roy. Soc. Lond. **A314**, 529 (1970).
[16] S. Weinberg, Gravitation and Cosmology, Wiley, New York (1972).
[17] R. M. Wald, General Relativity, University of Chicago Press (1984).
[18] T. Padmanabhan, Phys. Rept. **380**, 235 (2003).
[19] E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. **D15**, 1753 (2006).
[20] S. Nojiri and S. D. Odintsov, Phys. Rept. **505**, 59 (2011).
[21] S. Capozziello and M. De Laurentis, Phys. Rept. **509**, 167 (2011).

[22] L. Amendola and S. Tsujikawa, Dark Energy: Theory and Observations, Cambridge University Press (2010).
[23] V. Mukhanov, Physical Foundations of Cosmology, Cambridge University Press (2005).
[24] S. Dodelson, Modern Cosmology, Academic Press (2003).
[25] A. R. Liddle, An Introduction to Modern Cosmology, Wiley (2000).
[26] J. A. Peacock, Cosmological Physics, Cambridge University Press (1999).
[27] G. F. R. Ellis and R. Maartens, Class. Quant. Grav. **21**, 223 (2004).
[28] M. Novello and S. E. P. Bergliaffa, Phys. Rept. **463**, 127 (2008).
[29] Y. F. Cai et al., Phys. Rept. **493**, 1 (2010).
[30] U. Debnath et al., Class. Quant. Grav. **21**, 5609 (2004).
[31] A. Pradhan, J. P. Shahi and C. B. Singh, arXiv:gr-qc/0608070.
[32] G. P. Singh and K. Desikan, Pramana J. Phys. **49**, 205 (1997).
[33] J. K. Singh, Shaily, Shri Ram, J. R. L. Santos and J. A. S. Fortunato, arXiv:2209.06859.
[34] V. K. Bhardwaj and S. Prakash, Chinese J. Phys. (2023).
[35] D. M. Gusu, & M. V. Santhi, (2021). Analysis of Bianchi type V holographic dark energy models in general relativity and Lyra's geometry. Advances in High Energy Physics, (2021).