

On recurrent Light like Hypersurfaces of Indefinite Sasakian Manifold

Sushil Shukla

Department of Mathematics, Faculty of Engineering and Technology, Veer Bahadur Singh Purvanchal University, Jaunpur, India

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Abstract— The object of present paper is to study the properties of recurrent lightlike hypersurfaces of indefinite Sasakian manifold with (l, m) -type connection.

Keywords— Hypersurfaces, Recurrent lightlike hypersurfaces, Sasakian manifold.

I. INTRODUCTION

A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold (\bar{M}, \bar{g}) is called an (l, m) -type connection [1] if $\bar{\nabla}$ and its torsion tensor \bar{T} satisfy

$$(1.1) \quad (\bar{\nabla}_{\bar{X}} \bar{g})(\bar{Y}, \bar{Z}) = l\{\theta(\bar{Y})\bar{g}(\bar{X}, \bar{Z}) + \theta(\bar{Z})\bar{g}(\bar{X}, \bar{Y})\} - m\{\theta(\bar{Y})\bar{g}(J\bar{X}, \bar{Z}) + \theta(\bar{Z})\bar{g}(J\bar{X}, \bar{Y})\}$$

and

$$(1.2) \quad \bar{T}(\bar{X}, \bar{Y}) = l\{\theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}\} + m\{\theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}\}$$

where l and m are two smooth functions on \bar{M} , J is a tensor field of type $(1,1)$ and θ is a 1-form associated with a smooth unit vector field ζ which is called the characteristic vector field of \bar{M} , given by $\theta(\bar{X}) = \bar{g}(\bar{X}, \zeta)$.

By direct calculation it can be easily seen that a linear connection $\bar{\nabla}$ on M is an (l, m) -type connection if and only if $\bar{\nabla}$ satisfies

$$(1.3) \quad \bar{\nabla}_{\bar{X}} \bar{Y} = \nabla_{\bar{X}} \bar{Y} + \theta(\bar{Y})\{l\bar{X} + mJ\bar{X}\},$$

where ∇ is Levi-Civita connection of a semi-Riemannian manifold (\bar{M}, \bar{g}) with respect to \bar{g} .

In case $(l, m) = (1, 0)$: The above connection $\bar{\nabla}$ becomes a semi-symmetric non-metric connection. The notion of semisymmetric non-metric connection on a Riemannian manifold was introduced by Ageshe-Chafle [2,3] and later, studied by several authors [4,5]. In case $(l, m) = (0, 1)$: The above connection $\bar{\nabla}$ becomes a non-metric ϕ -symmetric connection such that $\phi(\bar{X}, \bar{Y}) = \bar{g}(J\bar{X}, \bar{Y})$. The notion of the non-metric ϕ -symmetric connection was introduced by Jin [6, 7, 8].

In case $(l, m) = (1, 0)$ in (1.1) and $(l, m) = (0, 1)$ in (1.2): The above connection $\bar{\nabla}$ becomes a quarter-symmetric non-metric connection. The notion of quarter-symmetric non-metric connection was introduced by Golab [9] and then, studied by Sengupta-Biswas [10] and Ahmad-Haseeb [11]. In case $(l, m) = (0, 0)$ in (1.1) and $(l, m) = (0, 1)$ in (1.2): The above connection $\bar{\nabla}$ becomes a quarter-symmetric metric connection. The notion of quarter-symmetric metric connection was introduced Yano-Imai [12]. In case $(l, m) = (0, 0)$ in (1.1) and $(l, m) = (1, 0)$ in (1.2): The above connection $\bar{\nabla}$ becomes a semi-symmetric metric connection. The notion of semi-symmetric metric connection was introduced by Hayden [13].

II. PRELIMINARIES

Let M be an almost contact manifold equipped with an almost contact metric structure $\{J, \zeta, \theta, \bar{g}\}$ consisting of

a (1,1) tensor field J , a vector field ζ , a 1-form θ and a compatible Riemannian metric \bar{g} satisfying

$$J^2 \bar{X} = -\bar{X} + \theta(\bar{X})\zeta, \quad \bar{g}(J\bar{X}, J\bar{Y}) \\ (2.1) \quad = \bar{g}(\bar{X}, \bar{Y}) - \theta(\bar{X})\theta(\bar{Y}), \theta(\zeta) = 1,$$

From this, we also have

$$J\zeta = 0, \quad \theta o J = 0, \quad \bar{g}(J\bar{X}, \bar{Y}) \\ = -\bar{g}(\bar{X}, J\bar{Y}), \quad \theta(\bar{X}) = \bar{g}(\bar{X}, \zeta).$$

for all $X, Y \in \chi(M)$.

An almost contact metric manifold M is a Sasakian manifold [14] if and only if it satisfies

$$(\bar{\nabla}_{\bar{X}} J)\bar{Y} = \bar{g}(\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})\bar{X}, \quad X, Y \in \chi(M).$$

where $\bar{\nabla}$ is Levi-Civita connection of the Riemannian metric \bar{g} .

With the above equation and (1.3), (2.1) and $\theta(JY) = 0$, it follows that

$$(\bar{\nabla}_{\bar{X}} J)\bar{Y} = \bar{g}(\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})\bar{X} \\ (2.2) \quad - \theta(\bar{Y})\{lJ\bar{X} - m\bar{X} + m\theta(\bar{X})\zeta\}.$$

Taking $\bar{Y} = \zeta$ and using $J\zeta = 0$ with $\theta(\bar{\nabla}_X \zeta) = l\theta(X)$, we have

$$(2.3) \quad \bar{\nabla}_{\bar{X}} \zeta = (m-1)J\bar{X} + l\bar{X},$$

Let (M, \bar{g}) be a lightlike hypersurface of \bar{M} . The normal bundle TM^\perp of M is a subbundle of the tangent bundle TM of M , of rank 1, and coincides with the radical distribution $\text{Rad}(TM) = TM \cap TM^\perp$. Denote by $F(M)$ the algebra of smooth functions on M and by $T(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M .

A complementary vector bundle $S(TM)$ of $\text{Rad}(TM)$ in TM is non-degenerate distribution on M , which is called a screen distribution on M , such that

$$TM = \text{Rad}(TM) \oplus_{\text{orth}} S(TM),$$

where \oplus_{orth} denotes the orthogonal direct sum. For any null section ξ of $\text{Rad}(TM)$, there exists a unique null section N of a unique lightlike vector bundle $\text{tr}(TM)$ in the orthogonal complement $S(TM)^\perp$ of $S(TM)$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N; N) = \bar{g}(N; X) = 0; \quad \forall X \in T(S(TM)).$$

We call $\text{tr}(TM)$ and N the transversal vector bundle and the null transversal vector field of M with respect to the screen distribution $S(TM)$, respectively.

The tangent bundle $T\bar{M}$ of \bar{M} is decomposed as follow:

$$T\bar{M} = TM \oplus \text{tr}(TM) = \{\text{Rad}(TM) \oplus \text{tr}(TM)\} \oplus_{\text{orth}} S(TM).$$

In the sequel, let X, Y, Z and W be the vector fields on M , unless otherwise specified. Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingarten formulas of M and $S(TM)$ are given respectively by

$$(2.4) \quad \bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N,$$

$$(2.5) \quad \bar{\nabla}_X N = -A_N X + \tau(X)N,$$

$$(2.6) \quad \nabla_X PY = \nabla_X^* PY + C(X, PY)\xi,$$

$$(2.7) \quad \nabla_X \xi = -A_\xi^* X - \sigma(X)\xi.$$

where $\bar{\nabla}$ and ∇^* are the induced linear connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_N and A_ξ^* are the shape operators on TM and $S(TM)$ respectively, and τ and σ are 1-forms on M .

For a lightlike hypersurface M of indefinite Sasakian manifold (\bar{M}, \bar{g}) , it is known [11] that $J(\text{Rad}(TM))$ and $J(\text{tr}(TM))$ are subbundles of $S(TM)$, of rank 1 such that $J(\text{Rad}(TM)) \cap J(\text{tr}(TM)) = 0$. Thus there exist two non-degenerate almost complex distributions D_o and D on M with respect to J , i.e., $J(D_o) = D_o$ and $J(D) = -D$, such that

$$S(TM) = J(\text{Rad}(TM)) \oplus J(\text{tr}(TM)) \oplus_{\text{orth}} D_o,$$

$$D = \{\text{Rad}(TM) \oplus_{\text{orth}} J(\text{Rad}(TM))\} \oplus_{\text{orth}} D_o,$$

$$TM = D \oplus J(\text{tr}(TM)).$$

Consider two null vector fields U and V such that

$$(2.8) \quad U = -JN, \quad V = -J\xi, \quad u(X) = g(X, V); \quad v(X) = g(X, U).$$

Denote by S the projection morphism of TM on D . Any vector field X of M is expressed as $X = SX + u(X)U$. Applying J to this form, we have

$$(2.9) \quad JX = FX + u(X)N,$$

where F is a tensor field of type $(1, 1)$ globally defined on M by $F = JoS$. Applying J to (2.9) and using (1.2), (1.3)

and (2.8), we have

$$(2.10) \quad F^2X = -X + u(X)U + \theta(X)\zeta.$$

As $u(U) = 1$ and $FU = 0$, the set (F, u, U) defines an indefinite almost contact structure on M and F is called the structure tensor field of M .

III. 3. (l, m)-type connections

Let (\bar{M}, \bar{g}, J) be Sasakian manifold with a semi-symmetric metric connection $\bar{\nabla}$. Using (1.1), (2.1) and (2.7), we obtain

$$(3.1) \quad (\bar{\nabla}_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y) - l\{\theta(Y)g(X, Z) + \theta(Z)g(X, Y)\} - m\{\theta(Y)g(JX, Z) + \theta(Z)g(JX, Y)\}$$

$$(3.2) \quad T(X, Y) = l\{\theta(Y)X - \theta(X)Y\} + m\{\theta(Y)FX - \theta(X)FY\}$$

$$(3.3) \quad B(X, Y) - B(Y, X) = m\{\theta(Y)u(X) - \theta(X)u(Y)\}$$

where T is the torsion tensor with respect to $\bar{\nabla}$ and η is a 1-form such that $\eta(X) = \bar{g}(X, N)$.

Proposition 1: Let M be a lightlike hypersurface of Sasakian manifold \bar{M} with an (l, m) -type connection such that ζ is tangent to M . Then B is symmetric if and only if $m = 0$.

Proof: If $m = 0$, then B is symmetric by (3.3). Conversely, if B is symmetric, then replacing X by ζ and Y by U , we get $m = 0$.

As $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \zeta)$, so B is independent of the choice of $S(TM)$ and satisfies

$$(3.4) \quad B(X, \zeta) = 0, \quad B(\zeta, X) = 0.$$

Local second fundamental forms are related to their shape operators by

$$(3.5) \quad B(X, Y) = g(A_\zeta^* X, Y) + mu(X)\theta(Y),$$

$$(3.6) \quad C(X, PY) = g(A_N X, PY) + \{l\eta(X) + mu(X)\}\theta(PY),$$

$$(3.7) \quad \bar{g}(A_\zeta^* X, N) = 0, \quad \bar{g}(A_N X, N) = 0, \quad \sigma = \tau.$$

$S(TM)$ is non-degenerate, so using (3.4)₂, (3.5), we have

$$A_\zeta^* \zeta = 0, \quad \bar{\nabla}_X \zeta = -A_\zeta^* X - \tau(X)\zeta.$$

Taking $\bar{\nabla}_X$ to $\bar{g}(\zeta, \zeta) = 0$ and $\bar{g}(\zeta, N) = 0$ and using (1.1), (2.3), (2.5), (3.5), (3.6) and (3.8), we have

$$(3.9) \quad g(A_\zeta^* X, \zeta) = -u(X), \quad B(X, \zeta) = (m-1)u(X). \\ (3.10) \quad g(A_N X, \zeta) = -v(X), \\ C(X, \zeta) = l\eta(X) + (m-1)v(X).$$

By (2.9), (2.3) and (2.4), we have

$$(3.11) \quad \bar{\nabla}_X \zeta = (m-1)FX + lX.$$

Applying $\bar{\nabla}_X$ to (2.8) and (2.9) and using (2.2), (2.4), (2.5), (2.9), (2.10), (3.1), (3.6), (3.8) with $\theta(U) = \theta(V) = 0$, we have

$$(3.12) \quad B(X, U) = C(X, V).$$

$$(3.13) \quad \bar{\nabla}_X U = F(A_N X) + \tau(X)U - \eta(X)\zeta,$$

$$(3.14) \quad \bar{\nabla}_X V = F(A_\zeta^* X) - \tau(X)V,$$

$$(3.15) \quad (\bar{\nabla}_X F)Y = u(Y)A_N X - B(X, Y)U + \{g(X, Y) - m\theta(X)\theta(Y)\}\zeta + (m-1)\theta(Y)X - l\theta(Y)FX,$$

$$(3.16)$$

$$(\bar{\nabla}_X u)Y = -u(Y)\tau(X) - B(X, FY) - l\theta(Y)u(X),$$

$$(3.17) \quad (\bar{\nabla}_X v)Y = v(Y)\tau(X) - g(A_N X, FY) - l\theta(Y)v(X) + (m-1)\theta(Y)\eta(X).$$

IV. RECURRENT HYPERSURFACES

Structure tensor field F of M is said to be recurrent [15] if there exists a 1-form ω on TM such that

$$(\bar{\nabla}_X F)Y = \omega(X)FY$$

A lightlike hypersurface M of Sasakian manifold \bar{M} is called recurrent if it admits a recurrent structure tensor field F .

Theorem 1: There exist no recurrent lightlike hypersurface of Sasakian manifold with an (l, m) -type connection such that ζ is tangent to M and F is recurrent.

Proof: As M is recurrent so by definition and (3.15), we have

$$(4.1) \quad w(X)FY = u(Y)A_N X - B(X, Y)U + \{g(X, Y) - m\theta(X)\theta(Y)\}\zeta + (m-1)\theta(Y)X - l\theta(Y)FX.$$

Taking $Y = \zeta$ and using (3.4) with $F\zeta = -V$,

$$w(X)V + u(X)\zeta = 0.$$

we have

Taking scalar product with U, we get $w=0$.

Hence F is parallel to ∇ .

Replacing Y by ζ and using (3.9), we get $(m-1)\{X - u(X)U - \theta(X)\zeta\} = lFX$.

Replacing X by V, we get $(m-1)V = l\zeta$, which implies $m=1$ and $l=0$.

Taking scalar product with ζ to (4.1) and using (3.10), we get $u(X)v(Y) - u(Y)v(X) = 0$.

Taking $X = U$ and $Y = V$, we have $m=0$, which is a contradiction that $(l, m) \neq (0, 0)$. Hence the theorem follows.

Corollary: There exist no recurrent lightlike hypersurface of Sasakian manifold with an (l, m) -type connection such that ζ is tangent to M and F is parallel with respect to connection ∇ of M.

V. LIE RECURRENT HYPERSURFACES

A Structure tensor field F of M is said to be Lie recurrent [15] if there exists a 1-form ν on TM such that

$$(L_X F)Y = \nu(X)FY.$$

where L_X denote the Lie derivative on M with respect to X.

Structure tensor field F is called Lie parallel if $L_X F = 0$.

A lightlike hypersurface M of Sasakian manifold \bar{M} is called Lie recurrent if it admits a Lie recurrent structure tensor field F.

Theorem 2: Let M be a Lie recurrent lightlike hypersurface of Sasakian manifold \bar{M} with an (l, m) -type connection such that ζ is tangent to M and F is Lie recurrent. Then

- (1) F is Lie parallel,

(2) 1-form τ satisfies $\tau = 0$ and

(3) Shape operator A_ζ^* satisfies $A_\zeta^*U = A_\zeta^*V = 0$.

Proof: (1) By definition of Lie recurrent, (2.9), (2.10), (3.2) and (3.15), we have

$$(4.2) \quad \mathcal{G}(X)FY = -\nabla_{FY}X + F\nabla_YX + u(Y)A_N X - \{B(X, Y) - m\theta(Y)u(X)\}U - \theta(Y)X + g(X, Y)\zeta.$$

Replacing Y by ζ and using (3.4), we have

$$(4.3) \quad -\nu(X)V = \nabla_V X + F\nabla_\zeta X.$$

Taking the scalar product with V and ζ , we get

$$(4.4) \quad u(\nabla_V X) = 0, \quad \theta(\nabla_V X) = 0.$$

Taking $Y = V$ in (4.2) and using $\theta(V) = 0$, we get

$$(4.5) \quad \nu(X)\zeta = -\nabla_\zeta X + F\nabla_V X - B(X, V)U + u(X)\zeta..$$

Applying F and using (2.10) and (4.4), we have

$$\nu(X)V = \nabla_V X + F\nabla_\zeta X.$$

Comparing this with (4.3), we have $\nu = 0$. Hence F is Lie parallel.

(2) Taking scalar product with N to (4.2) and using (3.7), we have

$$(4.6) \quad -\bar{g}(\nabla_{FY}X, N) + \bar{g}(\nabla_YX, U) = 0.$$

Taking $X = \zeta$ and using (2.7) with (3.5), we get

$$(4.7) \quad B(X, U) = \tau(FX)$$

Taking $X = U$ and using (3.12) with $FU = 0$, we have

$$(4.8) \quad C(U, V) = B(U, U) = 0.$$

Taking $X = V$ in (4.6) and using (3.5) with (3.14), we have

$$B(FY, U) = -\tau(Y).$$

Taking $Y = U$ and $Y = \zeta$ with the fact $FU = F\zeta = 0$, we get

$$(4.9) \quad \tau(U) = 0, \quad \tau(\zeta) = 0.$$

Taking $X = U$ to (4.2) and using (3.3), (3.10), (3.12), (3.13), we get

$$u(y)A_N U - F(A_N F Y) - A_N Y - \tau(F Y)U = 0.$$

Taking scalar product with V and using (3.6), (3.12) and (4.8), we get

$$B(X, U) = -\tau(FX).$$

Comparing with (4.70), we have $\tau(FX) = 0$.

Replacing X by FY and using (2.10) with (4.9), we have

$$\tau = 0.$$

(3) Taking $X = U$ in (3.3) and using (4.7) with $\tau = 0$, we have $B(U, X) = m\theta(X)$.

Taking $X = U$ in (3.5) and using (4.10), we have $(g(A_\xi^* U, X)) = 0$. Hence $A_\xi^* U = 0$

Replacing X by ξ in (4.3) and using (3.8) with $\tau = 0$, we have $A_N V = 0$.

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