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Existence of common fixed point for some generalized non-expansive mappings in *b*-metric space

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Abstract— *The concept of metric space or b-metric space plays a significant role in proving the existence theorem of solutions of differential equations or integral equations. A b-metric space is a generalization of a metric space. In this paper, we propose a generalization of non-expansive maps and investigate the conditions under which these maps have a common fixed point in b-metric spaces, and give examples illustrating our results. Our results are a generalization of some previous results.*
Keywords— ***fixed point, b-metric space, generalized non-expansive mapping, common fixed point***

I. INTRODUCTION

Since Banach's contraction principle has been of great application value, there has been a great deal of research to generalize it to date. The study of generalizing Banach's contraction principle is largely bifurcated. One is a generalization of the space structure in which the map is defined, and the other is a generalization of the contraction condition. Recently, there has been a growing effort to generalize the fixed point theorem for some contractions in b-metric spaces, which is a generalization of metric spaces (see e.g. [2]- [11]).

In [1], the fixed point theorem for generalized non-expansive mappings in complete metric space is introduced as follows.

Theorem1.1[2] *Let S , T map complete metric space X into itself and let a function family* $a_i(i=1,\dots,5)$ *map* $X\times X$ *into* $[0,1)$ *satisfying*

(1) $r=sup\{\sum_{i=1}^{5}a_i(x,y):x,y\in X\}<1$

(2) $a_2=a_3, a_4=a_5$

(3) $\forall x,y\in X$,

$$d(S(x),T(y))\le a_1d(x,y)+a_2d(x,T(y))+a_3d(y,S(x))+a_4d(x,S(x))+a_5d(y,T(y))$$

where $a_i=a_i(x,y)$. *Then either S or T has a fixed point. In particular if both S and T have fixed points, then S and T have unique fixed points respectively and these two fixed points coincide.*

The above fixed point theorem introduced in [1] is of significant importance as it is a generalization of different kinds of fixed point theorems. For instance, Banach's fixed point theorem is that if $S=T$ and $a_2=a_3=a_4=a_5=0$, then mapping T has a unique fixed point. Kannan proved in [10] that if $S=T$ and $a_1=a_2=a_3=0$, then mapping T has a unique fixed point and Reich proved in [11] that if $S=T$ and $a_2=a_3=0$, then mapping T has a unique fixed point. Hardy and Rodger proved in [7] that if $S=T$ then mapping T has a unique fixed point and Gupta and Srivastava proved in [5] that if $a_1=a_2=a_3=0$ and $a_4=a_5$, then S and T have a unique common fixed point.

As is shown above, the fixed point theorem introduced in [1] is of great importance as it includes various kinds of fixed point theorem as a special case. However, to date, we have not studied the condition for these maps to have fixed points in the b-metric space structure. Thus, generalizing this fixed point theorem to b-metric space, we can obtain a generalization of the above-mentioned several fixed point theorems in b-metric space, and we can study the fixed point theorems in b-metric space more extensively.

We generalize the contraction map introduced in [1] and present and prove a condition for these maps to have a unique common fixed point in a b-metric space. We also present results and examples that show that our results are a generalization of previous work.

II. MAIN RESULTS

The definition of a *b*-metric space is as follows.

Definition 2.1 ([6], [9]). Let X be a non-empty set and $s \geq 1$ be a given real number. d is called *b*-metric on X if the function $d: X \times X \to [0, \infty)$ satisfies

(1) $d(x, y) = 0 \iff x = y$

(2) $d(x, y) = d(y, x)$

(3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ (*b*-triangular inequality)

for all $x, y, z \in X$. (X, d) is called a *b*-metric space with a parameter *s*.

Theorem 2.1 *Let* S, T *be mappings with a parameter* $s \geq 1$ *that maps a complete b-metric space into itself and let function family* $a_i (i = 1, \ldots, 5)$ *map* $X \times X$ *into* $[0,1)$ *satisfying*

(1) $r := \sup\{a_1(x, y) + s a_2(x, y) + a_3(x, y) +$

$$+a_4(x, y) + a_5(x, y) : x, y \in X\} < 1$$

(2) $s a_2 = a_3, a_4 = a_5$

(3) $\forall x, y \in X,$

$$d\big(S(x), T(y)\big) \leq a_1 d(x, y) + a_2 d\big(x, T(y)\big) + a_3 d\big(y, S(x)\big) + a_4 d\big(x, S(x)\big) + a_5 d(y, T(y))$$

where $a_i = a_i(x, y)$. *Then either* S *or* T *has a fixed point. Especially if both* S *and* T *have fixed points then* S *and* T *have unique fixed points and they coincide each other.*

Proof. Let $x_0 \in X, x_{2n+1} = S(x_{2n}), x_{2n} = T(x_{2n+1}), n = 0,1,2$ .....

First let us prove that either S or T has a fixed point.

To this end, for an arbitrary n let us assume that $x_n \neq x_{n+1}$. If $x_n = x_{n+1}$ then it is proved straightforward. From the condition (3), we have

$$d(x_1, x_2) = d(S(x_0), T(x_1)) \leq a_1 d(x_0, x_1) + a_2 d(x_0, x_2) + a_3 d(x_1, x_1) + a_4 d(x_0, x_1) + a_5 d(x_1, x_2)$$
$$= (a_1 + a_4) d(x_0, x_1) + a_2 d(x_0, x_2) + a_5 d(x_1, x_2)$$

By *b*-triangular inequality we have

$$d(x_1, x_2) \leq s d(x_0, x_1) + s d(x_1, x_2)$$

And so

$$d(x_1, x_2) \leq (a_1 + s a_2 + a_4) d(x_0, x_1) + (s a_2 + a_5) d(x_1, x_2)$$

Thus we obtain

$$(1 - s a_2 - a_5) d(x_1, x_2) \leq$$
$$(a_1 + s a_2 + a_4) d(x_0, x_1) \quad (1)$$

Note that we have

$$\frac{a_1 + s a_2 + a_4}{1 - s a_2 - a_5} \leq \frac{r - a_3 - a_5}{1 - s a_2 - a_5} = \frac{r - a_3 - a_5}{1 - a_3 - a_5}$$

and by letting $a_3 + a_5 = x$, we clearly have $a_3 + a_5 \leq \frac{r}{2} < \frac{1}{2}$ and for arbitrary x within $[0, 1/2]$

$$\frac{r-x}{1-x} \le r$$

Accordingly we have $\frac{a_1+sa_2+a_4}{1-sa_2-a_5} \le r$ and by using (1) we obtain

$$d(x_1, x_2) \le rd(x_0, x_1)$$

Repeating this process gives

$$d(x_{n+2}, x_{n+1}) \le rd(x_{n+1}, x_n)$$

Therefore

$$d(x_{n+1}, x_n) \le r^n d(x_0, x_1)$$

Since $r<1$, we have

$$\sum_{n=0}^{\infty} d(x_{n+1}, x_n) < \infty$$

Thus we can conclude that sequence of points $\{x_n\}$ is cauchy sequence. By the completeness of (X, d), $\{x_n\}$ converges into the point x of X. Since $x_n \ne x_{n+1}$ for arbitrary n, either $x_{2n+1} \ne x$ or $x_{2n+1} \ne x$ is satisfied for infinitely many n. Without the loss of generality, assume that $x_{2n+1} \ne x$ for infinitely many n, i.e., there exist a subsequence $\{k(n)\}$ of $\{n\}$ such that there exist a sequence $\{x_{2k(n)+1}\}$ satisfying

$$x_{2k(n)+1} \ne x$$

for arbitrary n.

Then, we have

$$d(x, T(x)) \le sd(x, x_{2k(n)+1}) + sd\left(x_{2k(n)+1}, T(x)\right)$$
$$= sd(x, x_{2k(n)+1}) + sd(S(x_{2k(n)}), T(x)) \quad (2)$$

From (2) it holds that

$$d\left(S(x_{2k(n)}), T(x)\right) \le a_1 d(x_{2k(n)}, x)$$
$$+a_2 d\left(x_{2k(n)}, T(x)\right) + a_3 d(x, x_{2k(n)+1})$$
$$+a_4 d(x_{2k(n)}, x_{2k(n)+1}) + a_5 d(x, T(x))$$
$$\le d(x_{2k(n)}, x) + \frac{r}{2} d\left(x_{2k(n)}, T(x)\right) \quad (3)$$
$$+d(x, x_{2k(n)+1}) + d(x_{2k(n)}, x_{2k(n)+1}) + \frac{r}{2} d(x, T(x))$$

for $a_i = a_i(x, y)$ and taking the limit $n \to \infty$ in (2) and (3), we have

$$d(x, Tx) \le rd(x, Tx)$$

Since $r < 1$, we have $x = Tx$, i.e. T has unique fixed point.

Next, let x be the fixed point of S and y be the fixed point of T. Now we prove that $x = y$. If not, we can know that by (3), with respect to $a_i = a_i(x, y)$

$$d(x, y) = d(S(x), T(y)) \le (a_1 + sa_2 + a_3)d(x, y) < d(x, y)$$

must hold. This is contradictory and thus $x = y$. So S and T have a common fixed point. The uniqueness of the common fixed point is straightforward. □

Theorem 2.2 *Let T be a mapping from complete b-metric space (X, d) into itself. Assume that there exist a symmetric function family $a_1(x, y), a_4(x, y), a_5(x, y)$ that maps $X \times X$ into $[0, 1)$ satisfying*

1) $r = \sup\{(a_1(x, y) + a_4(x, y) + a_5(x, y))\} < 1$
2) $\forall x, y \in X$,

$$d(T(x), T(y)) \le a_1 d(x, y) + a_4 d(x, T(x)) + a_5 d(y, T(y))$$

where $a_i = a_i(x, y)$. *Then mapping* T *has a unique fixed point.*

Proof. For any $x, y \in X$, from the condition (2) we have

$$d(T(x), T(y)) \le a_1 d(x, y) + a_4 d(x, T(x)) + a_5 d(y, T(y))$$

$$d(T(y), T(x)) \le a_1 d(y, x) + a_4 d(y, T(y)) + a_5 d(x, T(x))$$

and therefore

$$d\big(T(x), T(y)\big) \le a_1 d(x, y) + \left(\frac{a_4 + a_5}{2}\right) d\big(x, T(x)\big)$$

$$+\left(\frac{a_4 + a_5}{2}\right) d(y, T(y)) \qquad (4)$$

Thus if we set

$$b_1 = a_1, b_2 = 0, b_3 = 0, b_4 = \frac{a_4 + a_5}{2}, b_5 = \frac{a_4 + a_5}{2}$$

then by (4) we obtain

$$d(T(x), T(y)) \le b_1 d(x, y) + b_2 d(x, T(y)) + b_3 d(y, T(x)) + b_4 d(x, T(x)) + b_5 d(y, T(y))$$

and

$$b_1 + s \cdot b_2 + b_3 + b_4 + b_5 = a_1 + a_4 + a_5 \le r$$

and the assumptions of theorem 1 are all satisfied. Therefore mapping T has a unique fixed point. □

Theorem 2.2 can be thought of as Reich's fixed point theorem in *b*-metric space.

Furthermore, setting $a_1 = 0$ in the above theorem gives us Kannan's fixed point theorem in *b*-metric space.

Example 1. Let $X = \{0, 1, 2\}$ and $d: X \times X \to [0, \infty)$ be defined as

$$d(0,1) = d(1, 0) = 1$$

$$d(0,2) = d(2, 0) = \frac{1}{5}$$

$$d(1,2) = d(2, 1) = \frac{3}{5}$$

$$d(0,0) = d(1, 1) = d(2, 2) = 0$$

It can be easily seen that metric space (X, d) is a complete *b*-metric space with the parameter $s = \frac{5}{4}$. Let us define the mapping $T: X \to X$ to be

$$T(0) = 0, T(1) = 2, T(2) = 0$$

and set $a_1 = a_4 = a_5 = \frac{1}{4}$. Then we have $a_1 + a_4 + a_5 = \frac{3}{4} < 1$ and

$$d\big(T(0), T(1)\big) = d(0, 2) = \frac{1}{5} \le \frac{1}{4} \cdot \left(1 + 0 + \frac{3}{5}\right) = a_1 d(0,1) + a_4 d\big(0, T(0)\big) + a_5 d(1, T(1))$$

$$d\big(T(0), T(2)\big) = d(0, 0) = 0 \le \frac{1}{4} \cdot \left(\frac{1}{5} + 0 + \frac{1}{5}\right) = a_1 d(0,2) + a_4 d\big(0, T(0)\big) + a_5 d(2, T(0))$$

$$d\big(T(1), T(2)\big) = d(2, 0) = \frac{1}{5} \le \frac{1}{4} \cdot \left(\frac{3}{5} + \frac{3}{5} + \frac{1}{5}\right) = a_1 d(1,2) + a_4 d\big(1, T(1)\big) + a_5 d(2, T(2))$$

Thus by theorem 2.2, T has a unique fixed point.

Example 2. Let $X = [0, 1]$ and $d: X \times X \to [0, \infty)$ be defined as $d(x, y) = |x - y|^2$. It is easily seen that (X, d) is a *b*-metric space with $s = 2$. For any $x \in X$, let us define T as $Tx := \frac{x}{4}$ and set $a_1 = a_4 = a_5 = \frac{1}{8}$. Then we have

$$a_1 + a_4 + a_5 = \frac{3}{8} < 1$$

Furthermore, for $\forall x, y \in X$, we obtain

$$d\big(T(x),T(y)\big)=\left|\frac{x}{4}-\frac{y}{4}\right|^2=\frac{1}{16}|x-y|^2\le\frac{1}{8}|x-y|^2+\frac{1}{8}\left|x-\frac{x}{4}\right|^2+\frac{1}{8}\left|y-\frac{y}{4}\right|^2$$
$$=a_1d(x,y)+a_4d(x,T(x))+a_5d(y,T(y))$$

Thus, from theorem 2.2, mapping T has a unique fixed point.

III. CONCLUSIONS

We generalized the condition of fixed point theorem introduced in [1] and presented and proved the condition for it to hold in *b*-metric space. Our results are an improvement and generalization of some previous fixed point results in complete metric space and *b*-metric space, such as [1], [4], [7].

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