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Stability analysis for nonlinear impulsive optimal control problems[1]

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Abstract—*We consider the generic stability of optimal control problems governed by nonlinear impulsive evolution equations. Under perturbations of the right-hand side functions of the controlled system, the results of stability for the impulsive optimal control problems are proved given set-valued theory.*

Keywords*— *Impulsive optimal control, Strongly continuous semigroup, PC mild solution, Stability, Set-valued mapping.

I. INTRODUCTION

Consider the problem:

Problem P: Find $\bar{u} \in U[0,T]$ such that

$$J(\bar{u}) = \min_{u \in U[0,T]} J(u), \tag{1.1}$$

where cost functional

$$J(u) = \int_0^T l(t, x(t), u(t))dt \tag{1.2}$$

subject to the nonlinear impulsive state equations

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), u(t)), t \in (0,T) \setminus D, \\ x(0) = x_0, \\ \Delta x(t_i) = J_i(x(t_i)), i = 1, 2, ..., N, \end{cases} \tag{1.3}$$

Where $T > 0$ is given, $x_0 \in X, x(t) \in X, u(t) \in U, U \subset E$ is compact, X and E are Banach spaces, $x_0 \in X$, $\Delta x(t_i) = x(t_i + 0) - x(t_i - 0) = x(t_i + 0) - x(t_i)$, $D = \{t_1, t_2, \ldots, t_N\} \subset (0,T)$, $0 < t_1 < t_2 < \cdots < t_N < T$,

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the $J_i(x(t_i))\ (i=1,2,\cdots,N)$ is a map which represents the size of jump, A is infinitesimal generator of strongly $T(\cdot)$, and the control set in the space continuous semigroup $L^1([0,T];E)$ is given by

$$U[0,T]\equiv\{u:[0,T]\to U\mid u(\cdot)\text{ is measurable function}\}. \tag{1.4}$$

Let $PC([0,T];X)=\{x:[0,T]\to X\mid x\text{ is continuous at } t\in[0,T]\setminus D,\ x\text{ is continuous from left and has right hand limits at } t\in D\}$. It is a Banach space equipped with the sup norm

$$\|x\|_{PC}=\sup_{t\in[0,T]}\|x(t)\|_X.$$

Let $B(x)$ be the Banach space of all linear and bounded operators on X. From [1], there exists a constant $M>0$ such that

$$\|T(t)\|_{B(x)}\le M \tag{1.5}$$

for all $t\in[0,T]$.

The following assumptions are imposed throughout the whole paper:

(H1) The function $f:[0,T]\times X\times U\to X$ is continuous concerning t and u. For any $x(t), y(t)\in X$, $u(t)\in U$, there exists a constant $L>0$ such that

$$\|f(t,x(t),u(t))-f(t,y(t),u(t))\|_X\le L\|x(t)-y(t)\|_X.$$

(**H2**) For all $x,y\in X$, there exist some constants $e_i\ge 0$ such that

$$\|J_i(x(t))-J_i(y(t))\|_X\le e_i\|x(t)-y(t)\|_X\ (i=1,2,\ldots,N),$$

and

$$M[LT+\sum_{i=1}^{N}e_i]<1.$$

(**H3**) The function $l:[0,T]\times X\times U\to R$ is continuous concerning t, x and u.

(**H4**) The control set $U[0,T]$ is compact in $L^1([0,T];\mathrm{E})$.

Remark 1.1 In many cases of optimal control problems, the control is piecewise continuous function, we need the control set

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is compact by setting some conditions about the set which is relatively compact in $L^1([0,T];\mathrm{E})$ from [13], namely, suppose that W is a subset of $L^1([0,T];\mathrm{E})$, denote $(\tau_h u)(t)=u(t+h)$ for $h>0$, W is relatively compact in $L^1([0,T];\mathrm{E})$ if and only if:

(1) The set $\{\int_{t_1}^{t_2} u(t)dt \mid u\in W\}$ is relatively compact in E, for any $0<t_1<t_2<T$;

(2) The limit $\|\tau_h u-u\|_{L^1([0,T-h];E)}\to 0$ as $h\to 0$, uniformly for $u\in W$.

Definition 1.1 For given $u\in U[0,T]$, x is called a PC mild solution of (1.3) if

$x\in PC([0,T];X)$ satisfies

$$x(t)=T(t)x_0+\int_0^t T(t-s)f(s,x(s),u(s))ds+\sum_{0<t_i<t} T(t-t_i)J_i(x(t_i)). \quad (1.6)$$

This solution is denoted by $x(\cdot,u(\cdot))$.

Applying assumptions (H1)-(H3) and the Theorem 2.1 of [8], we have the following Theorem.

Theorem 1.1 The system (1.3) has a unique PC mild solution.

Remark 1.2 we need to note that under some assumptions similarly to Remark 1.1, a set W in $PC([0,T];X)$ will be compact by generalized Ascoli-Arzela theorem from the Theorem 2.1 of [11]. We recall the result in the following from [11]:

Suppose $W\subseteq PC([0,T];X)$, if the following conditions are satisfied:

(1) W is a uniformly bounded subset of $PC([0,T];X)$

(2) W is equicotinuous in $I_i=(t_i,t_{i+1}), i=0,1,2,\dots,\mathrm{N}$, where $t_0=0, t_{N+1}=T$.

(3) Its t -sections $W(t)\equiv\{y(t)\mid y\in W, t\in[0,T]\setminus D\}, W(t_i+)\equiv\{y(t_i+)\mid y\in W\}$ and $W(t_i-)\equiv\{y(t_i-)\mid y\in W\}$ are relatively compact subsets of X.

Then W is a relatively compact subset of $PC([0,T];X)$. Furthermore, $PC([0,T];X)$ is continuously embedded in $L^1([0,T];E)$, then W is a relatively compact subset of $L^1([0,T];E)$.

In this paper, we consider the stability of optimal control problems governed by nonlinear impulsive evolution equation

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with respect to the right-hand side functions $f(t,x(t),u(t))$. we see some stability results in [7], [4] and [5] and there are many theories in impulsive systems such as [9], [3] and [10], to the best of our knowledge, under the case of multisolution optimal controls, the stability of optimal control problems governed by nonlinear impulsive evolution equation with respect to the right-hand side functions $f(t,x(t),u(t))$ have not been studied. For given fixed right-hand side functions $f(t,x(t),u(t))$, we can drive more than one optimal control, thus there exists a set mapping, we need to consider the stability of this set mapping.

2 Existence of impulsive optimal control problems

In this section, our first purpose is to consider the existence of optimal control and the solution is not unique, next is to show that the solution $x(\cdot,u(\cdot))$ of the system (1.3) is continuous concerning u .

We are recalling the Lemma 1.7.1 of [10] for impulsive integral inequality.

Lemma 2.1 For $t \ge t_0$ let a nonnegative piecewise continuous function $x(t)$ satisfy

$$x(t) \le c + \int_{t_0}^{t} v(s)x(s)ds + \sum_{t_0<\tau_i<t} b_i x(\tau_i) \tag{2.1}$$

where $c \ge 0, b_i \ge 0, v(s) > 0$. $x(t)$ has discontinuous points of the first kind at τ_i . Then we have

$$x(t) \le c \prod_{t_0<\tau_i<t} (1+b_i) e^{\int_{t_0}^{t} v(s)ds} . \tag{2.2}$$

Now we need the Lemma2.1 to consider the continuity of solution for system (1.3).

Theorem 2.1 Suppose assumptions (H1) and (H2) hold. $x \in PC([0,T];X)$ is the mild solution of system (1.3), then the map $u(\cdot) \to y(\cdot,u(\cdot))$ is continuous from $L^1([0,T];E)$ into $PC([0,T];X)$.

Proof. Denote

$$x_k(t) = T(t)x_0 + \int_0^t T(t-s) f(s,x_k(s),u_k(s))ds + \sum_{0<t_i<t} T(t-t_i) J_i(x_k(t_i))$$

and

$$x(t) = T(t)x_0 + \int_0^t T(t-s) f(s,x(s),u(s))ds + \sum_{0<t_i<t} T(t-t_i) J_i(x(t_i))$$

by the definition of PC mild solution. Applying the continuity of f and

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$$u_k \to u \text{ in } L^1(0,T;E)\ as\ k \to +\infty,$$

we get

$$\| f(t,x,u_k) - f(t,x,u)\|_X \to 0 \text{ for almost all } t \in [0,T], \text{ as } k \to +\infty,$$

and proceeding further, the limit

$$\lim_{k\to\infty} \int_0^T M \| f(s,x(s),u_k(s)) - f(s,x(s),u(s))\|_X ds = 0 \tag{2.3}$$

is hold.

According to inequality (1.5), it follows that

$$\begin{aligned}\|x_k(t)-x(t)\|_X &= \Big\| \int_0^t T(t-s)[f(s,x_k(s),u_k(s)) - f(s,x(s),u(s))]\mathrm{ds} \\ &+ \sum_{0<t_i<t} T(t-t_i)[J_i(x_k(t_i)) - J_i(x(t_i))] \Big\|_X \\ &\le \int_0^t M \|[f(s,x_k(s),u_k(s)) - f(s,x(s),u_k(s))]\|_X ds \\ &+ \int_0^T M \|[f(s,x(s),u_k(s)) - f(s,x(s),u(s))]\|_X ds \\ &+ \Big\| \sum_{0<t_i<t} T(t-t_i)[J_i(x_k(t_i)) - J_i(x(t_i))] \Big\|_X \\ &\le ML\int_0^t \|x_k(t)-x(t)\|_X ds + M\sum_{0<t_i<t} e_i \|x_k(t_i)-x(t_i)\|_X + \varepsilon\end{aligned},$$

and combining Lemma 2.1, we find

$$\|x_k(t)-x(t)\|_X \le \varepsilon e^{MLT} \prod_{0<t_i<t}(1+Me_i),\ \forall t\in[0,T]. \tag{2.4}$$

namely $x(\cdot,u_k) \to x(\cdot,u) \in PC([0,T];X)$. The proof is completed.

Next, the aim is to discuss the existence of optimal controls.

Theorem 2.2 The nonlinear impulsive optimal control problem P exist at least one optimal control.

Proof. Suppose $\{u_k\} \subseteq U[0,T]$ and

$$\lim_{k\to\infty} J(u_k) = \inf_{u\in U[0,T]} J(u). \tag{2.5}$$

It follows from the compactness of $U[0,T]$ that there exists a subsequence $\{u_{k'}\}$ of $\{u_k\}$ and $\bar{u} \in U[0,T]$ satisfy

$$u_{k'} \to \bar{u} \text{ in } L^1([0,T];E) \text{ as } k' \to +\infty, \tag{2.6}$$

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by the continuity of Theorem 2.1, we obtain

$$\left\| x_{u_{k'}}(t) - x_{\bar{u}}(t) \right\|_X \to 0 \ \text{a.e } t \in [0,T]$$

Combining compactness of E and continuity of l in assumptions, we see that there exists $M_1 > 0$ such that

$$\| l(t, x_{k'}(t), u_{k'}(t) \| \le M_1, \tag{2.7}$$

and

$$\lim_{k' \to +\infty} \int_0^T l(t, x_{k'}(t), u_{k'}(t))dt = \int_0^T l(t, x(t), \bar{u}(t))dt . \tag{2.8}$$

It is apparent from equation (2.8) that

$$\begin{aligned} J(\bar{u}) &= \int_0^T l(t, x(t), \bar{u}(t))dt \\ &= \int_0^T \lim_{k' \to +\infty} l(t, x_{k'}(t), u_{k'}(t))dt \\ &= \lim_{k' \to +\infty} \int_0^T l(t, x_{k'}(t), u_{k'}(t))dt \\ &= \inf_{u \in U[0,T]} J(u) \end{aligned}$$

This completes the proof of Theorem 2.2.

3 Stability analysis of impulsive optimal control problems

Under the definitions of upper semicontinuity, lower semicontinuity, compact upper semicontinuity, set-valued closed mapping, residual set, essential solution and generic stability from [7], [4], [5], [6], the purpose is to acquire the stability analysis of impulsive optimal control problems.

Setting

$$Y = \{ f \mid f \text{ satisfy the condition of (H1)} \},$$

in the controlled system, by [7], [4], [5], it is a complete metric space equipped with

the metric

$$d(f_1, f_2) = \sup_{(t,x,u) \in [0,T] \times X \times U} \| f_1(t,x,u) - f_2(t,x,u) \|_X$$

for every $f_1, f_2 \in Y$. Then the mapping $S : Y \to 2^{U[0,T]}$ is a set-valued mapping by

Theorem 2.2, where

$$S(f) = \{ \bar{u} \mid \bar{u} \text{ is the optimal control of problem } P \text{ associated with } f \in Y \}.$$

Suppose that (H1)-(H4) hold in the following theorems, from Theorem 2.2, we obtain the Theorem.

Theorem 3.1 $S(f) \ne \varnothing$ for each $f \in Y$.

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Theorem 3.2 $S:Y\to 2^{U[0,T]}$ is a compact upper semicontinuous mapping.

Proof. From the Lemma 4.1 of [5], we turn to show that

$$\text{Graph}(S)=\{(f,u)\in Y\times U[0,T]\,|\,u\in S(f)\}$$

is closed. Setting $f_k\to f$ in Y and $\bar{u}_k\to u_k$ in $L^1(0,T;E)$, then

$$J_{f_k}(\bar{u}_k)\le J_{f_k}(u),\text{ for all } u\in U[0,\mathrm{T}]. \tag{3.1}$$

From the Definition of PC mild solution, we can denote

$$x_f(t,\bar{u}(t))=T(t)x_0+\int_0^t T(t-s)f(s,x(s),\bar{u}(s))\,\mathrm{ds}+\sum_{0<t_i<t}T(t-t_i)J_i(x(t_i))$$

and

$$x_{f_k}(t,u_k(t))=T(t)x_0+\int_0^t T(t-s)f_k(s,x_k(s),u_k(s))\,\mathrm{ds}+\sum_{0<t_i<t}T(t-t_i)J_i(x_k(t_i))$$

for every $t\in[0,T]$.

It follows from $f_k\to f$ in Y that for any $\varepsilon>0$ there exists a constant $N_1>0$ such that

$$d(f_k,f)\le\frac{\varepsilon}{2MT}$$

whenever $k\ge N_1$. We have

$$\begin{aligned}&\int_0^T\|T(t-s)(f_k(s,x_k(s),u_k(s))-f(s,x_k(s),u_k(s)))\|_X ds\\ &\le\int_0^T M\,\|f_k(s,x_k(s),u_k(s))-f(s,x_k(s),u_k(s))\|_X ds\\ &\le\int_0^T M\,\frac{\varepsilon}{2MT}ds=\frac{\varepsilon}{2},\end{aligned} \tag{3.2}$$

imply that there exists a constant $N_2>0$ such that

$$\int_0^T M\,\|f(s,x(s),u_k(s))-f(s,x(s),\bar{u}(s))\|_X ds\le\frac{\varepsilon}{2} \tag{3.3}$$

whenever $k\ge N_2$.

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Taking $N = \max\{N_1, N_2\}$ and $k \geq N$, this leads to the following result

$$\begin{aligned}\left\|x_{f_k}(t,u_k(t)) - x_f(t,\bar{u}(t))\right\|_X &= \left\|\int_0^T (t-t_i)[f_k(s,x_k(s),u_k(s)) - f(s,x(s),\bar{u}(s))]\mathrm{ds}\right. \\ &\quad \left. + \sum_{0<t_i<t} T(t-t_i)[J_i(x_k(t_i) - J_i(x(t_i)]\right\|_X \\ &\leq \int_0^T M \left\| f_k(s,x_k(s),u_k(s)) - f(s,x_k(s),u_k(s))\right\|_X ds \\ &\quad + \int_0^T M \left\| f(s,x_k(s),u_k(s)) - f(s,x(s),u_k(s))\right\|_X ds \\ &\quad + \int_0^T M \left\| f(s,x(s),u_k(s)) - f(s,x(s),\bar{u}(s))\right\|_X ds \\ &\quad + \left\|\sum_{0<t_i<t} T(t-t_i)[J_i(x_k(t_i) - J_i(x(t_i))\right\|_X \\ &\leq ML\int_0^t \left\|x_k(t) - x(t)\right\|_X ds + \varepsilon \\ &\quad + M\sum_{0<t_i<t} e_i \left\|x_k(t_i) - x(t_i)\right\|_X\end{aligned}$$

and

$$\left\|x_{f_k}(t,u_k(t)) - x_f(t,\bar{u}(t))\right\|_X \leq \varepsilon e^{MLT} \prod_{0<t_i<t}(1+Me_i), \forall t \in [0,T]. \tag{3.4}$$

An argument similar to the one used in Theorem 2.2,

$$J_{f_k}(\bar{u}_k) \to J(\bar{u}), \tag{3.5}$$

also, we have

$$J_{f_k}(u) \to J_f(u), \text{ for all } u \in U[0,T]. \tag{3.6}$$

With the help of (3.1), (3.5) and (3.6), we can get

$$J_f(\bar{u}) \leq J_f(u), \text{ for all } u \in U[0,T], \tag{3.7}$$

which completes the proof.

Since Lemma 4.2 and Lemma 4.8 from [5], we obtain the result of generic stability of set-valued mapping $S(f)$ in the sense of Hausdorff metric.

Theorem 3.3 There exists a dense residual set $Q \subset Y$ such that $S(f)$ is stable in Q.

Remark 3.1 The results of essential approximation is also hold form Definition 3.1 and Theorem 3.3 of [7], and we only

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discuss the stability results about the nonlinear impulsive optimal control problem in this paper, if $e_i = 0\ (i = 1, 2, \ldots, N)$ in assumption (H4), then the system (1.3) become semilinear evolution system without impulsive, the results of stability and essential approximation are still valid.

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