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Derivation and Application of Multistep Methods to a Class of First-order Ordinary Differential Equations

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Abstract— *Of concern in this work is the derivation and implementation of the multistep methods through Taylor's expansion and numerical integration. For the Taylor's expansion method, the series is truncated after some terms to give the needed approximations which allows for the necessary substitutions for the derivatives to be evaluated on the differential equations. For the numerical integration technique, an interpolating polynomial that is determined by some data points replaces the differential equation function and it is integrated over a specified interval. The methods show that they are only convergent if and only if they are consistent and stable. In our numerical examples, the methods are applied on non-stiff initial value problems of first-order ordinary differential equations, where it is established that the multistep methods show superiority over the single-step methods in terms of robustness, efficiency, stability and accuracy, the only setback being that the multi-step methods require more computational effort than the single-step methods.*
Keywords— linear multi-step method; numerical solution; ordinary differential equation; initial value problem; stability; convergence.

I. INTRODUCTION

Linear multistep methods (LMMs) are very popular for solving initial value problems (IVPs) of ordinary differential equations (ODEs). They are also applied to solve higher order ODEs. LMMs are not self-starting hence, need starting values from single-step methods like Euler's method and Runge-Kutta family of methods.

The general k-step LMM is as given by Lambert [1]

$$\sum_{j=0}^{k}\alpha_j x_{n+j} = h\sum_{j=0}^{k}\beta_j \varphi_{n+j} \tag{1}$$

where α_j and β_j are uniquely determined and $\alpha_0 + \beta_0 \neq 0,\ \alpha_k = 1$. The LMM in Equation (1) generates discrete schemes which are used to solve first-order ODEs. Other researchers have introduced the continuous LMM using the continuous collocation and interpolation approach leading to the development of the continuous LMMs of the form

$$y(t) = \sum_{j=0}^{k}\alpha_j(t)x_{n+j} = h\sum_{j=0}^{k}\beta_j(t)\varphi_{n+j} \tag{2}$$

where α_j and β_j are expressed as continuous functions of x and are at least differentiable once [2].

According to [3], the existing methods of deriving the LMMs in discrete form include the interpolation approach, numerical integration, Taylor series expansion and through the determination of the order of LMM. Continuous collocation and interpolation technique are also used for the derivation of LMMs, block methods and hybrid methods.

In this study, we present the general multistep method, some of its different types and examine their characteristics. In light of this, we investigate the stability and convergence of these methods, compare the multistep methods with the single-step methods in operational time, accuracy and user-friendliness via some numerical examples.

In practice, only a few of the initial value differential equations that originate from the study of physical phenomena have exact solutions. The introduction however, of the multistep methods as numerical techniques is used in finding solutions to problems that have known exact solutions and in extension handle those problems whose exact solutions are not known. We shall limit this study to only non-stiff initial value problems of first-order ordinary differential equations.

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The Linear Multistep Methods

The general linear multistep method is given by

$$\sum_{j=0}^{k}\alpha_j x_{n+j} = h\sum_{j=0}^{k}\beta_j\varphi_{n+j} \tag{3}$$

where the α_j and β_j are constants, whereas $\alpha_k = 0$ and both α_0 and β_0 are not zero. Since (3) can be multiplied on both sides by the same constant without altering the relationship, the coefficients α_j and β_j are taken arbitrarily to the extent of a constant multiplier. In this work however, we will assume that $\alpha_k = 1$. If $\beta_k = 0$, the equation (3) is explicit otherwise it is implicit [4].

The Adams methods

These are the most important linear multistep methods for non-stiff initial value problems. It is the class of multistep methods (3) with $\alpha_k = 1$, $\alpha_{k-1} = -1$ and $\alpha_j = 0$, $j = 0,1,2,\cdots,k-2$. If equation (3) is given by

$$x_{n+k} = x_{n+k-1} + h\sum_{j=0}^{k-1}\beta_j\varphi_{n+j}, \tag{4}$$

then we have the Adams-Bashforth methods. And, if it is given by

$$x_{n+k} = x_{n+k-1} + h\sum_{j=0}^{k}\beta_j\varphi_{n+j}, \tag{5}$$

then we have the Adams-Moulton methods [5].

Predictor-Corrector (P-C) method

The multistep methods are often implemented in a 'predictor-corrector' form. In this way, a preliminary calculation is done using the explicit form of the multistep method then corrected using the implicit form of the multistep method. This is done by two calculations of the function φ at each step of this computation.

Order of linear multistep methods

We can associate the linear multistep method (3) with the linear difference operator θ, defined by

$$\theta\left[x(r);h\right] = \sum_{j=0}^{k}\left[\alpha_j x(r+jh) - h\beta_j x'(r+jh)\right], \tag{6}$$

where $x(r)$ is any arbitrary function that is continuously differentiable on the interval $[a,b]$. If the operator is allowed to operate on an arbitrary test function $x(r)$ with as many higher derivatives as we require, we can formally define the order of the operator and of the associated multistep method without invoking the solution of the initial value problem (4) which may possess only a first derivative. If we expand the test function $x(r+jh)$ and its derivative $x'(r+jh)$ as Taylor series about r and collect terms in (6) we have

$$\theta\left[x(r);h\right] = C_0 x(r) + C_1 h x'(r) + \cdots + C_q h^q x^{(q)}(r) + \cdots \tag{7}$$

where the C_q constants [1].

Definition 1 The difference operator (6) and the linear multistep method (3) associated with it are of order δ if, in (7), $C_0 = C_1 = C_2 = \cdots = C_\delta = 0$ and $C_{\delta+1} \neq 0$.

The following formulae for the constants C_q in terms of the coefficients α_j and β_j are given as:

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k \tag{8}$$

$$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + k\alpha_k - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k) \tag{9}$$

$$C_q = \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + \cdots + k^q\alpha_k - \frac{1}{(q-1)!(\beta_1 + 2^{q-1}\beta_2 + \cdots + k^{q-1}\beta_k)} \tag{10}$$

for $q = 2,3,\cdots$ [1].

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II. CHARACTERISTICS OF THE METHODS

With the number of approximations involved during computations using the multistep methods, the problem of consistency, stability and convergence call for discussion. The approximation in a one-step method depends directly on previous approximations alone, while the multistep method uses at least two of the previous approximations.

Consistency

The linear multistep method (3) is consistent if it has order $\delta \geq 1$[4]. From (8), (9) and (10), it follows that the method is consistent if and only if the following two conditions hold.

$$\sum_{j=0}^{k} \alpha_j = 0, \tag{11}$$

$$\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j \tag{12}$$

[1].

We shall subsequently consider the limits as $h \to 0$, $n \to \infty$ and $nh \to r - a$ remaining fixed.

Let x_n tend to $x(r)$ in the limit, that is $x_n \to x(r)$. Since k is fixed, we have that $x_{n+j} \to x(r)$, for $j = 0,1,2,\cdots,k$, or

$x(r) = x_{n+j} + \tau_{j,n}(h), \quad j = 0,1,2,\cdots,k$

where $\lim \tau_{j,n}(h) = 0, \quad j = 0,1,2,\cdots,k$ [1].

Hence, we have

$$\sum_{j=0}^{k} \alpha_j x(r) = \sum_{j=0}^{k} \alpha_j x_{n+j} + \sum_{j=0}^{k} \alpha_j \tau_{j,n}(h),$$

or replacing the first term on the right hand side of the equation by the term on the right hand side of (3) we have

$$x(r)\sum_{j=0}^{k} \alpha_j = h\sum_{j=0}^{k} \beta_j \varphi_{n+j} + \sum_{j=0}^{k} \alpha_j \tau_{j,n}(h).$$

In the limit, both terms on the right hand side vanish. Therefore, the left hand side becomes zero. The left hand side is not in general equal to zero, so we conclude that $\sum_{j=0}^{k} \alpha_j = 0$. The above argument holds if we merely assume that $\{x(r)\}$ tends to some function $x(r)$.

Condition (12) ensures that the function $x(r)$ does in fact satisfy the differential equation. For, under the limiting process,

$$\frac{x_{n+j} - x_n}{jh} \to x'(r), \qquad \text{for } j = 1,2,\cdots,k,$$

or, $x_{n+j} - x_n = jhx'(r) + jh\varphi_{j,n}(h), \quad$ for $j = 1,2,\cdots,k,$

where $\lim \varphi_{j,n}(h) = 0$. Hence,

$$\sum_{j=0}^{k} \alpha_j x_{n+j} - \sum_{j=0}^{k} \alpha_j x_n = h\sum_{j=0}^{k} j\alpha_j x'(r) + \sum_{j=0}^{k} \alpha_j \tau_{j,n}(h)$$

or

$$h\sum_{j=0}^{k} \beta_j \varphi_{n+j} - x_n \sum_{j=0}^{k} \alpha_j = hx'(r)\sum_{j=0}^{k} j\alpha_j + h\sum_{j=0}^{k} j\alpha_j \varphi_{j,n}(h).$$

Since $\sum_{j=0}^{k} \alpha_j = 0$, we have, on dividing through by h,

$$\sum_{j=0}^{k} \beta_j \varphi_{n+j} = x'(r)\sum_{j=0}^{k} j\alpha_j + \sum_{j=0}^{k} j\alpha_j \varphi_{j,n}(h)..$$

Under the limiting process, $\varphi_{n+j} \to \varphi(r, x(r))$, and, in the limit,

$$\varphi(r, x(r))\sum_{j=0}^{k} \beta_j = x'(r)\sum_{j=0}^{k} j\alpha_j..$$

Thus $x(r)$ satisfies the differential equation (4) if and only if $\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j$. This shows that if the sequence $\{x_n\}$ converges to the solution of the initial value problem (4) then the conditions (11) and (12) must hold [1].

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Stability

Let us introduce the first and second characteristic polynomials of the multistep method (3), defined as $\rho(\lambda)$ and $\sigma(\lambda)$ respectively, where $\rho(\lambda)=\sum_{j=0}^{k}\alpha_j\lambda^j$ and $\sigma(\lambda)=\sum_{j=0}^{k}\beta_j\lambda^j$ [6].

It follows from conditions (11) and (12) that the linear multistep method is consistent if and only if $\rho(1)=0$ and $\rho'(1)=\sigma(1)$ [1]. The stability of the multistep technique with respect to round-off error is clearly dictated by the magnitude of the zeros of the first polynomial above. However, the methods we have discussed in this work are zero-stable by virtue of their characteristics. The following are motivated by the types of zeros of the characteristic polynomial.

Theorem 3.1

Let λ_1, λ_2, …, λ_k denote the roots (real or complex), which may not necessarily be distinct, of the characteristic equation associated with the multistep difference method. If $|\lambda_n|\le 1$, for $n=1,2,\cdots,k$ and all the roots with absolute value equal to 1 are simple roots, then the difference method satisfies the root condition [7].

Theorem 3.2

i) The methods that satisfy the root condition with $\lambda=1$ as the only root of the characteristic equation with magnitude equal to 1 are said to be strongly stable. That is, the roots lie on the unit disc.

ii) If a method satisfies the root condition and has more than one distinct root with magnitude equal to 1, it is said to be weakly stable.

iii) If a method does not satisfy the root condition, it is said to be unstable. A multistep method is said to be stable if and only if it satisfies the root condition [7].

Convergence

One basic property that is demanded of an acceptable linear multistep technique is the convergence of the solution $\{x_n\}$ that is generated by the method, in some sense, to the theoretical solution $x(r)$ as the step-size *h* goes to zero. A linear multistep method is convergent if and only if it is consistent and stable, otherwise it is not convergent [8]. If a method is consistent but not stable, then it is not convergent. Also, if a method is stable but not consistent then it is not convergent.

Obtaining Starting Values

A multistep method is not self-starting, that is, a *k-step* multistep scheme requires some k previous values $x_0, x_1, x_2, ... x_{k-1}$. These *k* values that are needed to start the application of the multistep method are gotten by a single step method such as Taylor series method, Euler method or Runge-Kutta method. The starting method should be of the same or even lower order than the order of the multistep method itself.

Taylor series method

Let us consider the initial value problem

$$x'=\varphi(r,x),\quad x_0=\alpha. \tag{13}$$

Let us consider a numerical solution to (13) above using a k-step multistep method of order δ. We require that the starting values x_i, $i=1,2,\cdots,k-1$ should be calculated to an accuracy that is at least as high as the accuracy of the multistep method itself. That is, we require that $x_i-x(r_i)=O(h^{\delta+1})$, $i=1,2,\cdots,k-1$.

If enough partial derivatives of $\varphi(r,x)$ with respect to *r* and *x* exist, then we will use a truncated Taylor series to estimate x_i to any required degree of accuracy [9]. Thus, we have

$$x_{i+1}=x(r_i)+hx'(r_i)+\frac{h^2}{2!}x''(r_i)+\frac{h^3}{3!}x'''(r_i)+\cdots+O(h^{\delta+1}), \tag{14}$$

for $i=1,2,\cdots,k-1$. The derivatives in (14) are evaluated by successively differentiating the differential equation. Thus,

$$x(r_0)=x_0,$$

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$$x'(r)=\varphi(r,x),$$

$$x''(r)=\frac{\partial\varphi}{\partial r}+\frac{\partial\varphi}{\partial x}x'=\frac{\partial\varphi}{\partial r}+\varphi\frac{\partial\varphi}{\partial x}$$

$$x'''(r)=\frac{\partial^2\varphi}{\partial r^2}+2\varphi\frac{\partial^2\varphi}{\partial r\partial x}+\varphi^2\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial\varphi\partial\varphi}{\partial r\partial x}+\varphi\left(\frac{\partial\varphi}{\partial x}\right)^2.$$

This approach is theoretically flawless. Nevertheless, the evaluations of the total derivatives can be excessively tedious an d may not be adopted for an efficient computation.

Euler method

This is another method that can be employed to generate all the needed starting values for a linear multistep method.
Consider the equation

$$x'=\varphi(r,x),\quad a\le r\le b,\ x(a)=\alpha.$$

Let us suppose that the solution to the initial value problem (15) above has two continuous derivatives on the interval $[a,b]$, so that for each $i=1,2,\cdots,N-1$, we have

$$x_{i+1}=x(r_i)+(r_{i+1}-r_i)x'(r_i)+\frac{(r_{i+1}-r_i)^2}{2!}x''(\xi_i),$$

for some ξ_i in (r_i,r_{i+1}) . Since $h=r_{i+1}-r_i$, we have

$$x_{i+1}=x(r_i)+hx'(r_i)+\frac{h^2}{2!}x''(\xi_i)$$

and, since $x(r)$ satisfies our differential equation, we have

$$x_{i+1}=x(r_i)+h\varphi(r_i,x(r_i))+\frac{h^2}{2!}x''(\xi_i).$$

By deleting the remainder term, the Euler method becomes

$$x_0=\alpha$$

$$x_{i+1}=x_i+h\varphi(r,x_i),$$

for each $i=1,2,\cdots,N-1$.

The Euler method is gotten when the Taylor series method above is of order $\delta=1$. The simplicity of this method may be used to illustrate the techniques we intend to adopt in starting the multistep methods.

Runge-Kutta method

This method can also be applied to generate starting values for any multistep method.
We consider the equation

$$x(r+h)=x(r)+h\sum_{s=1}^{M}w_sk_s \tag{15}$$

where

$$k_1=\varphi(r,x),\quad k_s=\varphi(r+a_sh,x+h\sum_{i=1}^{s-1}\beta_{si}k_i,\ \text{ for } s=2,3,\cdots,M \text{ [1]}.$$

We call this an M-order Runge-Kutta method and it involves M function evaluations at each step. Each k_s, $k=1,2,\cdots,M$, may be interpreted as an approximation to the derivative $x'(r)$.

The objective is to choose w_s, a_s and β_{sr} so that the coefficients of h^i, $i=1,2,\cdots,M$, in equation (14) are identical with those of the equation (15). That is, the method must compare with the Taylor series method after its expansion. The higher order derivatives of the Taylor series expansion is given by

$$x'=\varphi$$

$$x''=\varphi_r+\varphi_x x'=\varphi_r+\varphi\varphi_x$$

$$x'''=\varphi_{rr}+2\varphi\varphi_{rx}+\varphi^2\varphi_{xx}+\varphi_x\varphi_r+\varphi\varphi_x{}^2$$

and so on.

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We note that the Runge-Kutta methods are not unique due to the manner in which they are derived. However, any Runge-Kutta methods of the same order are equivalent.

Runge-Kutta method of order two

This method uses two evaluations and it is given by

$$x_{i+1} = x_i + hw_i k_i + hw_2 k_2 ,$$

where $k_1 = \varphi(r_i, x_i)$, $k_2 = \varphi(r_i + \frac{1}{2}h, x_i + \frac{1}{2}hk_1)$

Runge-Kutta method of order four

This method uses four evaluations. It is given by

$$x_{i+1} = x_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4, \qquad (16)$$

Where $k_1 = h\varphi(r_i, x_i)$

$$k_2 = h\varphi(r_i + \frac{1}{2}h, x_i + \frac{1}{2}hk_1)$$

$$k_3 = h\varphi(r_i + \frac{h}{2}, x_i + \frac{h}{2}k_2)$$

$$k_4 = h\varphi(r_i + h, x_i + k_3)$$

[10].

We note however, that this is not unique. The Runge-Kutta method of order four shall be used in obtaining the starting values for the implementation of the multistep methods adopted in this work.

Derivations

Any specific linear multistep may be derived in a number of different ways. We shall consider a selection of different approaches which cast some light on the nature of the approximation involved.

Derivation through Taylor expansions

Euler method

Let us consider the Taylor series expansion for $x(r_n + h)$ about r_n,

$$x(r_n + h) = x(r_n) + hx'(r_n) + \frac{h^2}{2!}x''(r_n) + \cdots \qquad (17)$$

If we truncate this expansion after two terms and substitute for $x'(r)$ from the differential equation (4), we have

$$x(r_n + h) = x(r_n) + h\varphi(r_n, x(r_n)),$$

and the truncation error is $\frac{h^2}{2}x''(\xi_n)$ for ξ_n in (r_n, r_{n+1}).

If $x(r_n)$ and $x(r_n + h)$ are replaced by x_n and x_{n+1}, we get

$$x_{n+1} = x_n + h\varphi_n, \qquad (18)$$

which is an explicit linear one-step method [11]. This shall be used in solving the numerical examples in this work.

Mid-point rule

Let us consider the Taylor series expansions for $x(r_n + h)$ and $x(r_n - h)$ about r_n Thus,

$$x(r_n + h) = x(r_n) + hx'(r_n) + \frac{h^2}{2!}x''(r_n) + \frac{h^3}{3!}x'''(r_n) + \cdots$$

$$x(r_n - h) = x(r_n) - hx'(r_n) + \frac{h^2}{2!}x''(r_n) - \frac{h^3}{3!}x'''(r_n) + \cdots$$

Subtracting, we have

$$x(r_n + h) - x(r_n - h) = 2hx'(r_n) + \frac{h^3}{3!}x'''(r_n) + \cdots$$

Replacing $x(r_n + h)$ and $x(r_n - h)$ by x_{n+1} and x_{n-1}, we have $x_{n+1} - x_{n-1} = 2h\varphi_n$.

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This can be brought into the standard form of the linear multistep method (13), after replacing n by $n+1$, as

$$x_{n+2} - x_n = 2h\varphi_{n+1} \tag{19}$$

Its truncation error is $\pm\frac{1}{3}h^3 x'''(\xi_n)$ [1].

This is the mid-point rule and it shall be used in solving the numerical examples presented in this work.

The trapezoidal rule

If we wish to find the most accurate one-step implicit method $x_{n+1} + \alpha_0 x_n = h(\beta_1\varphi_{n+1} + \beta_0\varphi_n)$, we write down the associated approximate relationship

$$x(r_n + h) + \alpha_0 x(r_n) \approx h\left[\beta_1 x'(r_n + h) + \beta_0 x'(r_n)\right] \tag{20}$$

and choose α_0, β_0 and β_1 so as to make the approximation as accurate as possible.

The following expansions are used:

$$x(r_n + h) = x(r_n) + hx'(r_n) + \frac{h^2}{2!}x''(r_n) + \cdots$$

$$x'(r_n + h) = x'(r_n) + hx''(r_n) + \frac{h^2}{2!}x'''(r_n) + \cdots$$

Substituting these two equations into (20) and collecting the terms on the left-hand side gives

$$C_0 x(r_n) + C_1 hx'(r_n) + C_2 h^2 x''(r_n) + C_3 h^3 x'''(r_n) + \cdots \approx 0,$$

where

$$C_0 = 1 + \alpha_0,$$

$$C_1 = 1 - \beta_1 - \beta_0,$$

$$C_2 = \frac{1}{2} - \beta_1,$$

$$C_3 = \frac{1}{6} - \frac{1}{2}\beta_1.$$

Thus, in order to make the approximation in equation (20) as accurate as possible, we choose $\alpha_0 = -1$, $\beta_1 = \beta_0 = \frac{1}{2}$. The value of C_3 then becomes $-\frac{1}{12}$. The linear multistep method is now

$$x_{n+1} + x_n = \frac{h}{2}(\varphi_{n+1} + \varphi_n).$$

This is the trapezoidal rule and its local truncation error is $\pm\frac{1}{12}h^3 x'''(\xi_n)$ [1].

Derivation through numerical integration

This technique can be used to derive only a subclass of linear multistep methods consisting of those methods for which $\alpha_k = +1$, $\alpha_j = -1$, $\alpha_i = 0$, $i = 0,1,2,\cdots,j-1,j+1$, $j \neq k$. To start the derivation of any multistep method, we should note that the solution of the initial value problem given as

$$x' = \varphi(r,x), \quad a \le r \le b, \quad x(a) = \alpha,$$

if integrated over $[r_n, r_{n+1}]$, has the property that

$$x(r_{n+1}) - x(r_n) = \int_{r_n}^{r_{n+1}} x'(r)dr = \int_{r_n}^{r_{n+1}} \varphi(r,x(r))dr.$$

Consequently,

$$x(r_{n+1}) - x(r_n) = \int_{r_n}^{r_{n+1}} \varphi(r,x(r))dr. \tag{21}$$

We will integrate some interpolating polynomial $P(r)$ to $\varphi(r,x(r))$ which is determined by some previous data points that were obtained. Then, equation (21) becomes

$$x(r_{n+1}) - x(r_n) = \int_{r_n}^{r_{n+1}} P(r)dr. \tag{22}$$

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Simpson's rule

Suppose we want to derive a two-step method, we consider the identity

$$x(r_{n-2})-x(r_n)=\int_{r_n}^{r_{n+2}} x'(r)dr . \tag{23}$$

Using the differential equation (4), we can replace x' by $\varphi(r,x)$ in the integrand. The only available data for the approximate evaluation of the integral will be the values φ_n, φ_{n+1} and φ_{n+2}. Let $P(r)$ be the unique polynomial of second degree passing through the three points (r_n,φ_n), (r_{n+1},φ_{n+1}) and (r_{n+2},φ_{n+2}). By the Newton-Gregory forward interpolating formula,

$$P(r)=P(r_n+sh)=\varphi_n+s\Delta\varphi_n+\frac{s(s-1)}{2!}\Delta^2\varphi_n .$$

We now make the approximation

$$\int_{r_n}^{r_{n+2}} x'(r)dr \approx \int_{r_n}^{r_{n+2}} P(r)dr=\int_0^2[\varphi_n+s\Delta\varphi_n+\frac{1}{2}s(s-1)\Delta^2\varphi_n]hds$$

$$=h[2\varphi_n+2\Delta\varphi_n+\frac{1}{3}\Delta^2\varphi_n].$$

If we expand $\Delta\varphi_n$ and $\Delta^2\varphi_n$ in terms of φ_n, φ_{n+1}, φ_{n+2} and substitute in (23), we have

$$x_{n+2}-x_n=\frac{h}{3}[\varphi_{n+2}+4\varphi_{n+1}+\varphi_n] .$$

Then, the truncation error becomes $\pm\frac{1}{90}h^5x^{(5)}(\xi_n)$.

This is the Simpson's rule and it is the most accurate implicit linear two-step method [1].

Adams-Bashforth methods

Though any form of the interpolating polynomials could be used for the derivations, the Newton backward-difference formula will be used for the purpose of convenience. For us to derive an explicit k-step Adams-Bashforth method, we now form the backward-difference polynomial $P_{k-1}(r)$ through $(r_n,\varphi(r_n,x(r_n)))$, $(r_{n-1},\varphi(r_{n-1},x(r_{n-1})))$, $\cdots$, $(r_{n+1-k},\varphi(r_{n+1-k},x(r_{n+1-k})))$. Since $P_{k-1}(r)$ is an interpolating polynomial of degree $k-1$, then for some ξ_n in (r_{n+1-k},r_n), we have

$$\varphi(r,x(r)=P_{k-1}(r)+\frac{\varphi^{(k)}(\xi_n,x(\xi_n))}{k!}(r-r_n)(r-r_{n-1})\cdots(r-r_{n+1-k}) .$$

Introducing the substitution $r=r_n+sh$, and with $dr=hds$ into P_{k-1} and with the error term, it implies that

$$\int_{r_n}^{r_{n+1}}\varphi(r,x(r))dr=\int_{r_n}^{r_{n+1}}\sum_{m=0}^{k-1}(-1)^m\binom{-s}{m}\nabla^m\varphi(r_n,x(r_n))dr$$

$$+\int_{r_n}^{r_{n+1}}\frac{\varphi^{(k)}(\xi_n,x(\xi_n))}{k!}(r-r_n)(r-r_{n-1})\cdots(r-r_{n+1-k})dr$$

$$=\sum_{m=0}^{k-1}\nabla^m\varphi(r_n,x(r_n))h(-1)^m\int_0^1\binom{-s}{m}ds+\frac{h^{k+1}}{k!}\int_0^1 s(s+1)\cdots(s+k-1)\varphi^{(k)}(\xi_n,x(\xi_n))ds .$$

The integrals $(-1)^m\int_0^1\binom{-s}{m}ds$ for various values of m can be evaluated easily as displayed below.

For,

$$m=1:(-1)^m\int_0^1\binom{-s}{m}ds=\frac{1}{2}$$

$$m=2:(-1)^m\int_0^1\binom{-s}{m}ds=\frac{5}{12}$$

$$m=1:(-1)^m\int_0^1\binom{-s}{m}ds=\frac{3}{8}$$

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$$m=1:(-1)^m\int_0^1\binom{-s}{m}ds=\frac{251}{720}$$

$$m=1:(-1)^m\int_0^1\binom{-s}{m}ds=\frac{95}{288}$$

As a consequence of these evaluations,

$$\int_{r_n}^{r_{n+1}}\varphi(r,x(r))dr=h\left[\varphi(r_n,x(r_n))+\frac{1}{2}\nabla\varphi(r_n,x(r_n))+\frac{5}{12}\nabla^2\varphi(r_n,x(r_n))+\cdots\right]$$

$$+\frac{1}{k!}h^{k+1}\int_0^1 s(s+1)\cdots(s+1-k)\varphi^{(k)}(\xi_n,x(\xi_n))ds\,. \qquad (24)$$

The coefficient $s(s+1)(s+2)\cdots(s+k-1)$ does not change sign on the interval $[0,1]$ [12]. The weighted mean value theorem for integrals can be used to deduce that for a number μ_n, where $r_{n+1-k}<\mu_n<r_{n+1}$, the error term in equation (24) becomes

$$\frac{h^{k+1}}{k!}\int_0^1 s(s+1)\cdots(s+1-k)\varphi^{(k)}(\xi_n,x(\xi_n))ds=\frac{h^{k+1}\varphi^{k}(\mu_n,x(\mu_n))}{k!}\int_0^1 s(s+1)\cdots(s+k-1)ds$$

or the error term can be written as

$$h^{k+1}\varphi^{(k)}(\mu_n,x(\mu_n))(-1)^m\int_0^1\binom{-s}{k}ds$$

[7].

Adams-Bashforth two-step explicit method:

$$x(r_{n+1})\approx x(r_n)+h[\varphi(r_n,x(r_n))+\frac{1}{2}\nabla\varphi(r_n,x(r_n))]$$

$$=x(r_n)+h\varphi(r_n,x(r_n))+\frac{h}{2}[\varphi(r_n,x(r_n))-\varphi(r_{n-1},x(r_{n-1}))]$$

$$=x(r_n)+\frac{h}{2}[3\varphi(r_n,x(r_n))-\varphi(r_{n-1},x(r_{n-1}))].$$

Consequently, the explicit Adams-Bashforth two-step method is:

$$x_0=\alpha_0,\ x_1=\alpha_1$$

$$x_{n+1}=x_n+\frac{h}{2}[3\varphi(r_n,x(r_n))-\varphi(r_{n-1},x(r_{n-1}))]..$$

This can be taken into the standard form of the linear multistep method, after replacing n by $n+1$. Thus,

$$x_{n+2}=x_{n+1}+\frac{h}{2}[3\varphi_{n+1}-\varphi_n]. \qquad (25)$$

The local truncation error is

$$\tau_{n+1}(h)=\frac{5}{12}x'''(\mu_n)h^2\ ,$$

for some $\mu_n\in(r_{n-1},r_{n+1})$ [7].

This method shall be used in solving the numerical examples in this work.

Adams-Bashforth three-step explicit method:

$$x(r_{n+1})-x(r_n)\approx h\left[\varphi(r_n,x(r_n))+\frac{1}{2}\nabla\varphi(r_n,x(r_n))+\frac{5}{12}\nabla^2\varphi(r_n,x(r_n))\right]$$

$$=x(r_n)+h\left[\varphi(r_n,x(r_n))+\frac{1}{2}\{\varphi(r_n,x(r_n))-\varphi(r_{n-1},x(r_{n-1}))\}\right.$$

$$\left.+\frac{5}{12}\{\varphi(r_n,x(r_n))-2\varphi(r_{n-1},x(r_{n-1}))+\varphi(r_{n-2},x(r_{n-2}))\}\right]$$

$$=x(r_n)+\frac{h}{12}[23\varphi(r_n,x(r_n))-16\varphi(r_{n-1},x(r_{n-1}))+5\varphi(r_{n-2},x(r_{n-2}))].$$

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Consequently, the Adams-Bashforth three-step explicit method is

$$x_0 = \alpha_0,\ x_1 = \alpha_1,\ \ x_2 = \alpha_2$$

$$x_{n+1} = x_n + \frac{h}{12}\left[23\varphi(r_n, x_n) - 16\varphi(r_{n-1}, x_{n-1}) + 5\varphi(r_{n-2}, x_{n-2})\right],$$

for $n = 2, 3, \cdots, N-1$.

Replacing n by $n+2$, the standard form becomes

$$x_{n+3} = x_{n+2} + \frac{h}{12}\left[23\varphi_{n+2} - 16\varphi_{n+1} + 5\varphi_n\right]. \tag{26}$$

Again, the local truncation error can be shown to be

$$\tau_{n+1}(h) = \frac{3}{8}x^{(4)}(\mu_n)h^3, \text{ for } \mu_n \in (r_{n-2}, r_{n+1})\ [7].$$

Results from this method shall be displayed along with the results of the exact solutions for our numerical examples.

Adams-Bashforth four-step explicit method:

$$x_0 = \alpha_0,\ x_1 = \alpha_1,\ x_2 = \alpha_2,\ x_3 = \alpha_3$$

$$x_{n+1} = x_n + \frac{h}{24}\left[55\varphi(r_n, x_n) - 59\varphi(r_{n-1}, x_{n-1}) + 37\varphi(r_{n-2}, x_{n-2}) - 9\varphi(r_{n-3}, x_{n-3})\right],$$

where $n = 3, 4, \cdots, N-1$.

Replacing n by $n+3$, we have

$$x_{n+4} = x_{n+3} + \frac{h}{24}\left[55\varphi_{n+3} - 59\varphi_{n+2} + 37\varphi_{n+1} - 9\varphi_n\right]. \tag{27}$$

The local truncation error is

$$\tau_{n+1}(h) = \frac{251}{720}x^{(5)}(\mu_n)h^4, \text{ for some } \mu_n \in (r_{n-3}, r_{n+1})\ [7].$$

Results from this method shall be shown along with the results of the exact solutions for the numerical examples.

Adams-Bashforth five-step explicit method

$$x_0 = \alpha_0,\ x_1 = \alpha_1,\ x_2 = \alpha_2,\ x_3 = \alpha_3,\ x_4 = \alpha_4$$

$$x_{n+1} = x_n + \frac{h}{720}\begin{bmatrix} 1901\varphi(r_n, x_n) - 2774\varphi(r_{n-1}, x_{n-1}) + 2616\varphi(r_{n-2}, x_{n-2}) - \\ 1271\varphi(r_{n-3}, x_{n-3}) + 251\varphi(r_{n-4}, x_{n-4}) \end{bmatrix}$$

where $n = 4, 5, \cdots, N-1$.

Replacing n by $n+4$, we have

$$x_{n+5} = x_{n+4} + \frac{h}{720}\left[1901\varphi_{n+4} - 2774\varphi_{n+3} + 2616\varphi_{n+2} - 1271\varphi_{n+1} + 251\varphi_n\right].$$

The local truncation error is

$$\tau_{n+1}(h) = \frac{95}{288}x^{(6)}(\mu_n)h^5, \text{ for some } \mu_n \in (r_{n-4}, r_{n+1})\ [7].$$

Adams-Moulton method

To derive Adams-Moulton implicit *k-step* method, we can form the backward-difference polynomial $P_k(r)$ through $(r_{n+1}, \varphi(r_{n+1}, x(r_{n+1}))), (r_n, \varphi(r_n, x(r_n))), \cdots, (r_{n+1-k}, \varphi(r_{n+1-k}, x(r_{n+1-k})))$. Since $P(r)$ is an interpolating polynomial of degree k, then for some ξ_n in (r_{n+1-k}, r_{n+1}), we have,

$$\varphi(r, x(r)) = P_k(r) + \frac{\varphi^{(k+1)}(\xi_n, x(\xi_n))}{(k+1)!}(r - r_{n+1})(r - r_n)\cdots(r - r_{n+1-k}).$$

Introducing the substitution $r = r_n + sh$, and with $dr = hds$ into P_k and with the error term, it implies that

$$\int_{r_n}^{r_{n+1}} \varphi(r, x(r))dr = \int_{r_n}^{r_{n+1}} \sum_{m=0}^{k}(-1)^m \binom{-s}{m} \nabla^m \varphi(r_n, x(r_n))dr$$

$$+\int_{r_n}^{r_{n+1}}\frac{\varphi^{(k+1)}(\xi_n,x(\xi_n))}{(k+1)!}(r-r_{n+1})(r-r_n)\cdots(r-r_{n+1-k})dr$$

$$=\sum_{m=0}^{k}\nabla^m\varphi(r_n,x(r_n))h(-1)^m\int_0^1\binom{-s}{m}ds$$

$$+\frac{h^{k+2}}{(k+1)!}\int_0^1(s-1)s(s+1)\cdots(s+k-1)\varphi^{(k+1)}(\xi_n,x(\xi_n))ds.$$

The integrals $(-1)^m\int_0^1\binom{-s}{m}ds$ for various values of m can be evaluated easily as displayed below:

For,

$$m=1:(-1)^m\int_0^1\binom{-s}{m}ds=-\frac{1}{2}$$

$$m=2:(-1)^m\int_0^1\binom{-s}{m}ds=-\frac{1}{12}$$

$$m=1:(-1)^m\int_0^1\binom{-s}{m}ds=-\frac{1}{24}$$

$$m=1:(-1)^m\int_0^1\binom{-s}{m}ds=-\frac{19}{720}$$

$$m=1:(-1)^m\int_0^1\binom{-s}{m}ds=-\frac{3}{160}.$$

As a consequence of these evaluations,

$$\int_{r_n}^{r_{n+1}}\varphi(r,x(r))dr=h\left[\varphi(r_{n+1},x(r_{n+1}))-\frac{1}{2}\nabla\varphi(r_{n+1},x(r_{n+1}))-\frac{1}{12}\nabla^2\varphi(r_{n+1},x(r_{n+1}))+\cdots\right]$$

$$+\frac{h^{k+2}}{(k+1)!}\int_0^1(s-1)s(s+1)\cdots(s+1-k)\varphi^{(k+1)}(\xi_n,x(\xi_n))ds.$$

The coefficient $(s-1)s(s+1)(s+2)\cdots(s+k-1)$ does not change sign on the interval $[0,1]$ [12]. The weighted mean value theorem for integrals can be used to deduce that for a number μ_n, where $r_{n+1-k}<\mu_n<r_{n+1}$, the error term in equation above becomes

$$\frac{h^{k+2}}{(k+1)!}\int_0^1(s-1)s(s+1)\cdots(s+1-k)\varphi^{(k+1)}(\xi_n,x(\xi_n))ds$$

$$=\frac{h^{(k+2)}\varphi^{(k+1)}(\mu_n,x(\mu_n))}{(k+1)!}\int_0^1(s-1)s(s+1)\cdots(s+k-1)ds,$$

or the error term can be written as

$$h^{k+2}\varphi^{(k+1)}(\mu_n,x(\mu_n))(-1)^m\int_0^1\binom{-s}{k}ds$$

[13].

Adams-Moulton two-step implicit method:

$$x_0=\alpha_0,\ x_1=\alpha_1$$

$$x_{n+1}=x_n+\frac{h}{12}\left[5\varphi(r_{n+1},x_{n+1})+8\varphi(r_n,x_n)-\varphi(r_n,x_n)\right]$$

where $n=1,2,\cdots,N-1.$

To put this in standard form, we replace n by $n+1$. Thus,

$$x_{n+2}=x_{n+1}+\frac{h}{12}\left[5\varphi(r_{n+2},x_{n+2})+8\varphi(r_{n+1},x_{n+1})-\varphi(r_n,x_n)\right].$$

The local truncation error is

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$$\tau_{n+1}(h) = -\frac{1}{24}x^{(4)}(\mu_n)h^3, \text{ for some } \mu_n \in (r_{n-1}, r_{n+1}) \text{ [7].}$$

Adams-Moulton three-step method:

$$x_0 = \alpha_0,\ x_1 = \alpha_1,\ x_2 = \alpha_2,$$

$$x_{n+1} = x_n + \frac{h}{24}\left[9\varphi(r_{n+1}, x_{n+1}) + 19\varphi(r_n, x_n) - 5\varphi(r_{n-1}, x_{n-1}) + \varphi(r_{n-2}, x_{n-2})\right],$$

for $n = 2, 3, \cdots, N-1$.

Replacing n by $n+2$, we have

$$x_{n+3} = x_{n+2} + \frac{h}{24}\left[9\varphi(r_{n+3}, x_{n+3}) + 19\varphi(r_{n+2}, x_{n+2}) - 5\varphi(r_{n+1}, x_{n+1}) + \varphi(r_n, x_n)\right]$$

The local truncation is

$$\tau_{n+1}(h) = -\frac{19}{720}x^{(5)}(\mu_n)h^4, \text{ for } \mu_n \in (r_{n-2}, r_{n+1}) \text{ [7].}$$

Adams-Bashforth four-step explicit method:

$$x_0 = \alpha_0,\ x_1 = \alpha_1,\ x_2 = \alpha_2,\ x_3 = \alpha_3,$$

$$x_{n+1} = x_n + \frac{h}{720}\left[251\varphi(r_{n+1}, x_{n+1}) + 646\varphi(r_n, x_n) - 264\varphi(r_{n-1}, x_{n-1}) + 106\varphi(r_{n-2}, x_{n-2}) - 19\varphi(r_{n-3}, x_{n-3})\right], \text{ where}$$

$n = 3, 4, \cdots, N-1$.

Replacing n by $n+3$, we have

$$x_{n+4} = x_{n+3} + \frac{h}{720}\left[251\varphi(r_{n+4}, x_{n+4}) + 646\varphi(r_{n+3}, x_{n+3}) - 264\varphi(r_{n+2}, x_{n+2}) + 106\varphi(r_{n+1}, x_{n+1}) - 19\varphi(r_n, x_n)\right]$$

The local truncation error is

$$\tau_{n+1}(h) = -\frac{3}{160}x^{(6)}(\mu_n)h^5, \text{ for some } \mu_n \in (r_{n-3}, r_{n+1}) \text{ [7].}$$

Numerical Examples

We will now solve the following problems using some of the methods discussed in this work and the results displayed in tables along with the results of the corresponding exact solutions.

Example1: Given that

$$x' = x - r^2, \quad x(0) = 1,$$

we obtain the values of x for $r = 0.1, 0.2, 0.3, 0.4$ by the Euler method, Runge-Kutta method of order four, mid-point rule, Adams-Bashforth two-step explicit method, Adams-Bashforth three-step explicit method and Adams-Bashforth four-step explicit method and then compare the result with the exact solution $x(r) = r^2 + 2r + 2 - e^r$ to obtain the error.

Example 2: We solve the initial value problem

$xx' = r$, $\quad x(0) = 1$, $\quad 0 \le r \le 0.4$,

using the Euler method, Runge-Kutta method of order four, mid-point rule, Adams-Bashforth two-step explicit method, Adams-Bashforth three-step explicit method and Adams-Bashforth four-step explicit method with step size $h = 0.1$ and then compare the results with the exact solution $x(r) = \sqrt{r^2 + 1}$.

Euler method for Example 1

Using equation (18) $x_{n+1} = x_n + h\varphi_n$,

$n = 0:$ $\quad x_1 = x_0 + h\varphi_0$

$$r_0 = 0,\quad x_0 = 1$$

$$\varphi_0 = \varphi(0,1) = 1$$

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$x_1 = 1 + 0.1 \times 1 = 1.1$

$n = 1:$ $x_2 = x_1 + h\varphi_1$

$r_1 = 0.1, \quad x_1 = 1.1$

$\varphi_1 = \varphi(0.1, 1.1) = 1.09$

$x_2 = 1 + 0.1 \times 1.09 = 1.209$

$n = 2:$ $x_3 = x_2 + h\varphi_2$

$r_2 = 0.2, \quad x_2 = 1.209$

$\varphi_2 = \varphi(0.2, 1.209) = 1.169$

$x_3 = 1.209 + 0.1 \times 1.169 = 1.3259$

$n = 3:$ $x_4 = x_3 + h\varphi_3$

$r_3 = 0.3, \quad x_3 = 1.3259$

$\varphi_3 = \varphi(0.3, 1.3259) = 1.2359$

$x_4 = 1.3259 + 0.1 \times 1.2359 = 1.449949$

The result of the Euler method is displayed in TABLE1.

Table.1: Euler's rule for Example 1

r	$x(r)$	Exact solution	Error
0.0	1.000000	1.000000	0.000000
0.1	1.100000	1.104829	0.004829
0.2	1.209000	1.218597	0.009597
0.3	1.325900	1.340141	0.014241
0.4	1.449949	1.468175	0.018226

Runge-Kutta method of order four for Example 1

Using equation (16) $x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$,

where $k_1 = h\varphi(r_n, x_n)$,

$$k_2 = h\varphi\left(r_n + \frac{h}{2}, x_n + \frac{1}{2}k_1\right)$$

$$k_3 = h\varphi\left(r_n + \frac{h}{2}, x_n + \frac{1}{2}k_2\right)$$

$$k_4 = h\varphi(r_n + h, x_n + k_3)$$

$n = 0:$ $r_0 = 0, \quad x_0 = 1$

$k_1 = 0.1 \times \varphi(0,1)$

$\varphi(0,1) = 1$

$k_1 = 0.1 \times 1 = 0.1$

$k_2 = 0.1 \times \varphi(0.05, 1.05)$

$\varphi(0.05, 1.05) = 0.10475$

$k_2 = 0.1 \times 0.10475 = 0.10475$

$k_3 = 0.1 \times \varphi(0.05, 1.052375)$

$\varphi(0.05, 1.052375) = 1.049875$

$k_3 = 0.1 \times 1.049875 = 0.1049875$

$k_4 = 0.1 \times \varphi(0.1, 1.1049875)$

$\varphi(0.1, 1.1049875) = 1.0949875$

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$$k_4 = 0.1\times 1.0949875 = 0.10949875 \; x_1 = 1+\frac{1}{6}\left[0.1+2(0.10475)+2(0.1049875)+0.10949875\right] = 1.104829$$

$n=1:$ $\quad r_1 = 0.1, \quad x_1 = 1.104829$

$$k_1 = 0.1\times\varphi(0.1,1.104829)$$
$$\varphi(0.1,1.104829) = 1.094829$$
$$k_1 = 0.1\times 1.104829 = 0.109483$$
$$k_2 = 0.1\times\varphi(0.15,1.159571)$$
$$\varphi(0.15,1.159571) = 1.137071$$
$$k_2 = 0.1\times 1.137071 = 0.113707$$
$$k_3 = 0.1\times\varphi(0.15,1.161683)$$
$$\varphi(0.15,1.161183) = 1.139183$$
$$k_3 = 0.1\times 1.139183 = 0.113918$$
$$k_4 = 0.1\times\varphi(0.2,1.218747)$$
$$\varphi(0.2,1.218747) = 1.178747$$
$$k_4 = 0.1\times 1.178747 = 0.117875$$
$$x_2 = 1.104829+\frac{1}{6}\left[0.109483+2(0.113707)+2(0.113918)+0.117875\right] = 1.218597$$

$n=2:$ $\quad r_2 = 0.2, \quad x_2 = 1.218597$

$$k_1 = 0.1\times\varphi(0.2,1.218597)$$
$$\varphi(0.2,1.218597) = 1.178597$$
$$k_1 = 0.1\times 1.178597 = 0.117860$$
$$k_2 = 0.1\times\varphi(0.25,1.277527)$$
$$\varphi(0.25,1.277527) = 1.215027$$
$$k_2 = 0.1\times 1.215027 = 0.121503$$
$$k_3 = 0.1\times\varphi(0.25,1.279349)$$
$$\varphi(0.25,1.279349) = 1.216849$$
$$k_3 = 0.1\times 1.216849 = 0.121685$$
$$k_4 = 0.1\times\varphi(0.3,1.340282)$$
$$\varphi(0.3,1.340282) = 1.250282$$
$$k_4 = 0.1\times 1.250282 = 0.125028$$
$$x_3 = 1.218597+\frac{1}{6}\left[0.117860+2(0.121503)+2(0.121685)+0.125028\right] = 1.340141$$

$n=3:$ $\quad r_3 = 0.3, \quad x_1 = 1.340141$

$$k_1 = 0.1\times\varphi(0.3,1.340141)$$
$$\varphi(0.3,1.340141) = 1.250141$$
$$k_1 = 0.1\times 1.250141 = 0.125014$$
$$k_2 = 0.1\times\varphi(0.35,1.402648)$$
$$\varphi(0.35,1.402648) = 1.280148$$
$$k_2 = 0.1\times 1.280148 = 0.128015$$
$$k_3 = 0.1\times\varphi(0.35,1.404149)$$

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$$\varphi(0.35,1.404149) = 1.281649$$

$$k_3 = 0.1 \times 1.281649 = 0.128165$$

$$k_4 = 0.1 \times \varphi(0.4,1.468306)$$

$$\varphi(0.4,1.468306) = 1.308306$$

$$k_4 = 0.1 \times 1.308306 = 0.130831$$

$$x_4 = 1.340141 + \frac{1}{6}\left[0.125014 + 2(0.128015) + 2(0.128165) + 0.130831\right] = 1.468175$$

The result of the Runge-Kutta method is displayed on TABLE2.

Table.2: Runge-Kutta method of order four for Example 1

r	$x(r)$	Exact solution	Error
0.0	1.000000	1.000000	0.000000
0.1	1.104829	1.104829	0.000000
0.2	1.218597	1.218597	0.000000
0.3	1.340141	1.340141	0.000000
0.4	1.468175	1.468175	0.000000

Mid-point rule for Example 1

Using equation (19) $x_{n+2} = x_n + 2h\varphi_{n+1}$,

$n = 0:$ $\quad x_2 = x_0 + 2h\varphi_1$

$$r_0 = 0, \quad x_0 = 1, \quad r_1 = 0.1, \quad x_1 = 1.104829$$

$$\varphi_1 = \varphi(0.1,1.104829) = 1.094829$$

$$x_2 = 1 + 2 \times 0.1 \times 1.094829 = 1.218966$$

$n = 1:$ $\quad x_3 = x_1 + 2h\varphi_2$

$$r_2 = 0.2, \quad x_2 = 1.218966$$

$$\varphi_2 = \varphi(0.2,1.218966) = 1.178966$$

$$x_3 = 1.104829 + 2 \times 0.1 \times 1.178966 = 1.340622$$

$n = 2:$ $\quad x_4 = x_2 + 2h\varphi_3$

$$r_3 = 0.3, \quad x_3 = 1.340622$$

$$\varphi_3 = \varphi(0.3,1.340622) = 1.250622$$

$$x_4 = 1.218966 + 2 \times 0.1 \times 1.250622 = 1.6909$$

The result of the mid-point rule is displayed on TABLE 3.

Table.3. Mid-point rule for Example 1

r	$x(r)$	Exact solution	Error
0.0	1.000000	1.000000	0.000000
0.1	1.104829	1.104829	0.000000
0.2	1.218966	1.218597	0.000369
0.3	1.340622	1.340141	0.000481
0.4	1.469090	1.468175	0.000915

Adams-Bashforth two-step explicit method for Example 1

Using equation (25) $x_{n+2} = x_{n+1} + \frac{h}{2}[3\varphi_{n+1} - \varphi_n]$,

$n = 0:$ $\quad x_2 = x_1 + \frac{h}{2}[3\varphi_1 - \varphi_0]$

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$$r_0 = 0, \quad x_0 = 1, \quad r_1 = 0.1, \quad x_1 = 1.104829$$

$$\varphi_0 = \varphi(0,1) = 1$$

$$\varphi_1 = \varphi(0.1, 1.104829) = 1.094829$$

$$x_2 = 1.104829 + \frac{0.1}{2}\left[3(1.094829) - 1\right] = 1.219053$$

$$n = 1: \quad x_3 = x_2 + \frac{h}{2}[3\varphi_2 - \varphi_1]$$

$$r_1 = 0.1, \quad x_1 = 1.104829, \quad r_2 = 0.2, \quad x_2 = 1.219053$$

$$\varphi_1 = \varphi(0.1, 1.104829) = 1.094829$$

$$\varphi_2 = (0.2, 1.219053) = 1.179053$$

$$x_2 = 1.219053 + \frac{0.1}{2}\left[3(1.179053) - 1.094829\right] = 1.341170$$

$$n = 2: \quad x_4 = x_3 + \frac{h}{2}[3\varphi_3 - \varphi_2]$$

$$r_2 = 0.2, \; x_2 = 1.219053, \; r_3 = 0.3, \quad x_3 = 1.341170$$

$$\varphi_2 = \varphi(0.2, 1.219053) = 1.179053$$

$$\varphi_3 = (0.3, 1.341170) = 1.251170$$

$$x_4 = 1.341170 + \frac{0.1}{2}\left[3(1.251170) - 1.179053\right] = 1.469893$$

The result of the Adams-Bashforth two-step method is displayed in TABLE 4.

Table.4: Adams-Bashforth two-step method for Example 1

r	*x(r)*	Exact solution	Error
0.0	1.000000	1.000000	0.000000
0.1	1.104829	1.104829	0.000000
0.2	1.219053	1.218597	0.000456
0.3	1.341170	1.340141	0.001029
0.4	1.469893	1.468175	0.001718

Adams-Bashforth three-step explicit method for Example 1

Using equation (26) $\quad x_{n+3} = x_{n+2} + \frac{h}{12}\left[23\varphi_{n+2} - 16\varphi_{n+1} + 5\varphi_n\right]$

$$n = 0: \quad x_3 = x_2 + \frac{h}{12}\left[23\varphi_2 - 16\varphi_1 + 5\varphi_0\right]$$

$$r_0 = 0, \; x_0 = 1, \; r_1 = 0.1, \; x_1 = 1.104829, \; r_2 = 0.2,$$

$$x_2 = 1.218597$$

$$\varphi_0 = \varphi(0,1) = 1$$

$$\varphi_1 = \varphi(0.1, 1.104829) = 1.094829$$

$$\varphi_2 = \varphi(0.2, 1.218597) = 1.178597$$

$$x_3 = 1.218597 + \frac{0.1}{12}\left[23(1.178597) - 16(1.094829) + 5(1)\right] = 1.340184$$

$$n = 1: \quad x_4 = x_3 + \frac{h}{12}\left[23\varphi_3 - 16\varphi_2 + 5\varphi_1\right]$$

$$r_1 = 0.1, \; x_1 = 1.104829, \; r_2 = 0.2, \; x_2 = 1.218597$$

$$r_3 = 0.3, \; x_3 = 1.340141$$

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$$\varphi_1 = \varphi(0.1, 1.104829) = 1.094829$$

$$\varphi_2 = \varphi(0.2, 1.218597) = 1.178597$$

$$\varphi_3 = \varphi(0.3, 1.340184) = 1.250184$$

$$x_4 = 1.340184 + \frac{0.1}{12}\left[23(1.250184) - 16(1.178597) + 5(1.094829)\right] = 1.468274$$

The result of the Adams-Bashforth three-step explicit method is displayed in TABLE 5.

Table.5: Adams-Bashforth three-step explicit method for Example 1

r	x(r)	Exact solution	Error
0.0	1.000000	1.000000	0.000000
0.1	1.104829	1.104829	0.000000
0.2	1.218597	1.218597	0.000000
0.3	1.340184	1.340141	0.000043
0.4	1.468274	1.468175	0.000099

Adams-Bashforth four-step explicit method for Example 1

Using equation (27) $x_{n+4} = x_{n+3} + \frac{h}{24}\left[55\varphi_{n+3} - 59\varphi_{n+2} + 37\varphi_{n+1} - 9\varphi_n\right]$

$n = 0:$ $x_4 = x_3 + \frac{h}{24}\left[55\varphi_3 - 59\varphi_2 + 37\varphi_1 - 9\varphi_0\right]$

$r_0 = 0,\ x_0 = 1,\ r_1 = 0.1,\ x_1 = 1.104829,\ r_2 = 0.2,$

$x_2 = 1.218597,\ r_3 = 0.3,\ x_3 = 1.340141,$

$$\varphi_0 = \varphi(0, 1) = 1$$

$$\varphi_1 = \varphi(0.1, 1.104829) = 1.094829$$

$$\varphi_2 = \varphi(0.2, 1.218597) = 1.178597$$

$$\varphi_3 = \varphi(0.3, 1.340141) = 1.250141$$

$$x_4 = 1.340141 + \frac{0.1}{24}\left[55(1.250141) - 59(1.178597) + 37(1.094829) - 9(1)\right] = 1.468179.$$

The result of the Adams-Bashforth four-step explicit method is displayed in TABLE 6.

Table.6: Adams-Bashforth four-step explicit method for Example 1

r	x(r)	Exact solution	Error
0.0	1.000000	1.000000	0.000000
0.1	1.104829	1.104829	0.000000
0.2	1.218597	1.218597	0.000000
0.3	1.340141	1.340141	0.000000
0.4	1.468179	1.468175	0.000004

Euler method for Example 2

Using equation (18) $x_{n+1} = x_n + h\varphi_n,$

$n = 0:$ $x_1 = x_0 + h\varphi_0$

$r_0 = 0,\ \ x_0 = 1$

$\varphi_0 = \varphi(0, 1) = 0$

$x_1 = 1 + 0.1 \times 0 = 1$

$n = 1:$ $x_2 = x_1 + h\varphi_1$

$r_1 = 0.1,\ \ x_1 = 1$

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$\varphi_1 = \varphi(0.1,1) = 0.1$

$x_2 = 1 + 0.1 \times 0.1 = 1.01$

$n = 2:$ $x_3 = x_2 + h\varphi_2$

$r_2 = 0.2$, $x_2 = 1.01$

$\varphi_2 = \varphi(0.2,1.01) = 0.198020$

$x_3 = 1.01 + 0.1 \times 0.198020 = 1.029802$

$n = 3:$ $x_4 = x_3 + h\varphi_3$

$r_3 = 0.3$, $x_3 = 1.029802$

$\varphi_3 = \varphi(0.3,1.029802) = 0.291318$

$x_4 = 1.029802 + 0.1 \times 0.291318 = 1.058934$

The result of the Euler method is displayed in TABLE 7.

Table.7: Euler's rule for Example 2

r	*x(r)*	Exact solution	Error
0.0	1.000000	1.000000	0.000000
0.1	1.000000	1.104988	0.004988
0.2	1.010000	1.019804	0.009804
0.3	1.029802	1.044031	0.014229
0.4	1.058934	1.077033	0.018099

Runge-Kutta method of order four for Example 2

Using equation (16) $x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$,

where $k_1 = h\varphi(r_n, x_n)$,

$$k_2 = h\varphi\left(r_n + \frac{h}{2}, x_n + \frac{1}{2}k_1\right)$$

$$k_3 = h\varphi\left(r_n + \frac{h}{2}, x_n + \frac{1}{2}k_2\right)$$

$$k_4 = h\varphi(r_n + h, x_n + k_3)$$

$r_0 = 0$, $x_0 = 1$

$k_1 = 0.1 \times \varphi(0,1)$

$\varphi(0,1) = 0$

$k_1 = 0.1 \times 0 = 0$

$k_2 = 0.1 \times \varphi(0.05,1)$

$\varphi(0.05,1) = 0.05$

$k_2 = 0.1 \times 0.05 = 0.005$

$k_3 = 0.1 \times \varphi(0.05,1.0025)$

$\varphi(0.05,1.0025) = 0.049875$

$k_3 = 0.1 \times 0.049875 = 0.004988$

$k_4 = 0.1 \times \varphi(0.1,1.004988)$

$\varphi(0.1,1.004988) = 0.099504$

$k_4 = 0.1 \times 0.009950 = 0.009950$ $x_1 = 1 + \frac{1}{6}\left[0 + 2(0.005) + 2(0.004988) + 0.009950\right] = 1.004988$

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$n = 1:$ $r_1 = 0.1, \quad x_1 = 1.004988$

$$k_1 = 0.1 \times \varphi(0.1, 1.004988)$$
$$\varphi(0.1, 1.004988) = 0.099504$$
$$k_1 = 0.1 \times 0.099504 = 0.009950$$
$$k_2 = 0.1 \times \varphi(0.15, 1.009963)$$
$$\varphi(0.15, 1.009963) = 0.148520$$
$$k_2 = 0.1 \times 0.148520 = 0.014852$$
$$k_3 = 0.1 \times \varphi(0.15, 1.0112414)$$
$$\varphi(0.15, 1.012414) = 0.148161$$
$$k_3 = 0.1 \times 0.148161 = 0.014816$$
$$k_4 = 0.1 \times \varphi(0.2, 1.019804)$$
$$\varphi(0.2, 1.019804) = 0.196116$$
$$k_4 = 0.1 \times 0.196116 = 0.019612$$
$$x_2 = 1.04988 + \frac{1}{6}\left[0.009950 + 2(0.014852) + 2(0.014816) + 0.019612\right] = 1.019804$$

$n = 2:$ $r_2 = 0.2, \quad x_2 = 1.019804$

$$k_1 = 0.1 \times \varphi(0.2, 1.019804)$$
$$\varphi(0.2, 1.019804) = 0.196116$$
$$k_1 = 0.1 \times 0.196116 = 0.019612$$
$$k_2 = 0.1 \times \varphi(0.25, 1.029610)$$
$$\varphi(0.25, 1.029610) = 0.242810$$
$$k_2 = 0.1 \times 0.242810 = 0.024281$$
$$k_3 = 0.1 \times \varphi(0.25, 1.031945)$$
$$\varphi(0.25, 1.031945) = 0.242261$$
$$k_3 = 0.1 \times 0.242261 = 0.024226$$
$$k_4 = 0.1 \times \varphi(0.3, 1.044030)$$
$$\varphi(0.3, 1.044030) = 0.287348$$
$$k_4 = 0.1 \times 0.287348 = 0.028735$$
$$x_3 = 1.019804 + \frac{1}{6}\left[0.019612 + 2(0.024281) + 2(0.024226) + 0.028735\right] = 1.044031$$

$n = 3$ $r_3 = 0.3, \quad x_1 = 1.044031$

$$k_1 = 0.1 \times \varphi(0.3, 1.044031)$$
$$\varphi(0.3, 1.044031) = 0.287348$$
$$k_1 = 0.1 \times 0.287348 = 0.028735$$
$$k_2 = 0.1 \times \varphi(0.35, 1.058399)$$
$$\varphi(0.35, 1.058399) = 0.330688$$
$$k_2 = 0.1 \times 0.330688 = 0.033069$$
$$k_3 = 0.1 \times \varphi(0.35, 1.060566)$$
$$\varphi(0.35, 1.060566) = 0.330012$$
$$k_3 = 0.1 \times 0.330012 = 0.033001$$

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$$k_4 = 0.1 \times \varphi(0.4, 1.077032)$$

$$\varphi(0.4, 1.077032) = 0.371391$$

$$k_4 = 0.1 \times 0.371391 = 0.037139$$

$$x_4 = 1.044031 + \frac{1}{6}\left[0.028735 + 2(0.033069) + 2(0.033001) + 0.037139\right] = 1.077033.$$

The result of the Runke-Kutta method is displayed in TABLE 8.

Table.8: Runge-Kutta method of order four for Example 2

r	*x(r)*	Exact solution	Error
0.0	1.000000	1.000000	0.000000
0.1	1.004988	1.004988	0.000000
0.2	1.019804	1.019804	0.000000
0.3	1.044031	1.044031	0.000000
0.4	1.077033	1.077033	0.000000

Mid-point rule for Example 2

Using equation (19) $x_{n+2} = x_n + 2h\varphi_{n+1}$,

$n = 0:$ $x_2 = x_0 + 2h\varphi_1$

$r_0 = 0$, $x_0 = 1$, $r_1 = 0.1$, $x_1 = 1.004988$

$\varphi_1 = \varphi(0.1, 1.004988) = 0.99504$

$x_2 = 1 + 2 \times 0.1 \times 0.099504 = 1.019901$

$n = 1:$ $x_3 = x_1 + 2h\varphi_2$

$r_2 = 0.2$, $x_2 = 1.019901$

$\varphi_2 = \varphi(0.2, 1.019901) = 0.196097$

$x_3 = 1.004988 + 2 \times 0.1 \times 0.196097 = 1.044207$

$n = 2:$ $x_4 = x_2 + 2h\varphi_3$

$r_3 = 0.3$, $x_3 = 1.044207$

$\varphi_3 = \varphi(0.3, 1.044207) = 0.287299$

$x_4 = 1.019901 + 2 \times 0.1 \times 0.287299 = 1.077361$

The result of the mid-point rule is displayed in TABLE 9.

Table.9: Mid-point rule for Example 2

r	*x(r)*	Exact solution	Error
0.0	1.000000	1.000000	0.000000
0.1	1.004988	1.004988	0.000000
0.2	1.019901	1.019804	0.000097
0.3	1.044207	1.044031	0.000176
0.4	1.077361	1.077033	0.000328

Adams-Bashforth two-step explicit method for Example 2

Using equation (25) $x_{n+2} = x_{n+1} + \frac{h}{2}\left[3\varphi_{n+1} - \varphi_n\right]$,

$n = 0:$ $x_2 = x_1 + \frac{h}{2}\left[3\varphi_1 - \varphi_0\right]$

$r_0 = 0$, $x_0 = 1$, $r_1 = 0.1$, $x_1 = 1.004988$

$\varphi_0 = \varphi(0, 1) = 0$

$$\varphi_1 = \varphi(0.1, 1.004988) = 0.099504$$

$$x_2 = 1.004988 + \frac{0.1}{2}\left[3(0.099504) - 0\right] = 1.019914$$

$n = 1:$ $$x_3 = x_2 + \frac{h}{2}[3\varphi_2 - \varphi_1]$$

$$r_1 = 0.1, \quad x_1 = 1.004988, \quad r_2 = 0.2, \quad x_2 = 1.019914$$

$$\varphi_1 = \varphi(0.1, 1.004988) = 0.099504$$

$$\varphi_2 = (0.2, 1.019914) = 0.196095$$

$$x_2 = 1.019914 + \frac{0.1}{2}\left[3(0.196095) - 0.099504\right] = 1.044353$$

$n = 2:$ $$x_4 = x_3 + \frac{h}{2}[3\varphi_3 - \varphi_2]$$

$$r_2 = 0.2, \quad x_2 = 1.019914, \quad r_3 = 0.3, \quad x_3 = 1.044353$$

$$\varphi_2 = \varphi(0.2, 1.019914) = 0.196095$$

$$\varphi_3 = (0.3, 1.044353) = 0.287259$$

$$x_4 = 1.044353 + \frac{0.1}{2}\left[3(0.287259) - 0.196095\right] = 1.077637$$

The result of the Adams-Bashforth two-step explicit method is displayed in Table 10.

Table.10: Adams-Bashforth two-step explicit method for Example 2

r	*x(r)*	Exact solution	Error
0.0	1.000000	1.000000	0.000000
0.1	1.004988	1.004988	0.000000
0.2	1.019914	1.019804	0.000110
0.3	1.044353	1.044031	0.000322
0.4	1.077637	1.077033	0.000604

Adams-Bashforth three-step explicit method for Example 2

Using equation (26) $$x_{n+3} = x_{n+2} + \frac{h}{12}\left[23\varphi_{n+2} - 16\varphi_{n+1} + 5\varphi_n\right]$$

$n = 0:$ $$x_3 = x_2 + \frac{h}{12}\left[23\varphi_2 - 16\varphi_1 + 5\varphi_0\right]$$

$$r_0 = 0, \; x_0 = 1, \; r_1 = 0.1, \; x_1 = 1.004988, \; r_2 = 0.2,$$

$$x_2 = 1.019804$$

$$\varphi_0 = \varphi(0, 1) = 0$$

$$\varphi_1 = \varphi(0.1, 1.004988) = 0.099504$$

$$\varphi_2 = \varphi(0.2, 1.01904) = 0.196116$$

$$x_3 = 1.019804 + \frac{0.1}{12}\left[23(0.196116) - 16(0.099504) + 5(0)\right] = 1.044126$$

$n = 1:$ $$x_4 = x_3 + \frac{h}{12}\left[23\varphi_3 - 16\varphi_2 + 5\varphi_1\right]$$

$$r_1 = 0.1, \; x_1 = 1.004988, \; r_2 = 0.2, \; x_2 = 1.019804$$

$$r_3 = 0.3, \; x_3 = 1.044126$$

$$\varphi_1 = \varphi(0.1, 1.004988) = 0.099504$$

$$\varphi_2 = \varphi(0.2, 1.019804) = 0.196116$$

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$$\varphi_3 = \varphi(0.3, 1.044126) = 0.287322$$

$$x_4 = 1.044126 + \frac{0.1}{12}\left[23(0.287322) - 16(0.196116) + 5(0.099504)\right] = 1.077193$$

The result of the Adams-Bashforth three-step method is displayed in TABLE 11.

Table.11: Adams-Bashforth three-step explicit method for Example 2

r	$x(r)$	Exact solution	Error
0.0	1.000000	1.000000	0.000000
0.1	1.004988	1.004988	0.000000
0.2	1.019804	1.019804	0.000000
0.3	1.044126	1.044031	0.000095
0.4	1.077193	1.077033	0.000160

Adams-Bashforth four-step explicit method for Example 2

Using equation (27) $x_{n+4} = x_{n+3} + \frac{h}{24}\left[55\varphi_{n+3} - 59\varphi_{n+2} + 37\varphi_{n+1} - 9\varphi_n\right]$

$$n = 0: \quad x_4 = x_3 + \frac{h}{24}\left[55\varphi_3 - 59\varphi_2 + 37\varphi_1 - 9\varphi_0\right]$$

$r_0 = 0$, $x_0 = 1$, $r_1 = 0.1$, $x_1 = 1.004988$, $r_2 = 0.2$,

$x_2 = 1.019804$, $r_3 = 0.3$, $x_3 = 1.044031$,

$$\varphi_0 = \varphi(0,1) = 0$$

$$\varphi_1 = \varphi(0.1, 1.004988) = 0.099504$$

$$\varphi_2 = \varphi(0.2, 1.019804) = 0.196116$$

$$\varphi_3 = \varphi(0.3, 1.044031) = 0.287348$$

$$x_4 = 1.044031 + \frac{0.1}{24}\left[55(0.287348) - 59(0.196116) + 37(0.099504) - 9(0)\right] = 1.077010$$

The result of the Adams-Bashforth four-step method is displayed in TABLE 12.

Table.12: Adams-Bashforth four-step explicit method for Example 2

R	$x(r)$	Exact solution	Error
0.0	1.000000	1.000000	0.000000
0.1	1.004988	1.004988	0.000000
0.2	1.019804	1.019804	0.000000
0.3	1.044031	1.044031	0.000000
0.4	1.077010	1.077033	0.000023

III. CONCLUSION

In this study, it is seen that the multistep methods are derived using Taylor series expansion and numerical integration. The numerical integration approach uses the interpolatory polynomial which is determined by some data points to approximate the solution to the differential equations.Here, we employed different single-step and multistep schemes in solving non-stiff initial value problems of ordinary differential equations. We found out that,unlike the single-step methods, the multistep methods attempt to gain efficiency by using information from all previously computed steps to compute the next solution value. From the results of our numerical examples, it is established that the multistep methods, though involving more computational effort, clearly show superiority in terms of accuracy compared to the single-step methods. The linear multistep methods being discussedalso show stability, and hence ensure convergence.Consistency is seen to hold for the multistep methods since they have orders that are greater than or equal to 1 (i.e. $\lambda \geq 1$).

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