International Journal of Chemistry, Mathematics and Physics (IJCMP)
[Vol-6, Issue-4, Jul-Aug, 2022]
https://dx.doi.org/10.22161/ijcmp.6.4.1
ISSN: 2456-866X

The Khalimsky Line Topology- Countability and Connectedness

S.A Bhuiyan

Department of CSE, Leading University, Sylhet, Bangladesh
Email: aktersabia@yahoo.com

Received: 25 Jun 2022; Received in revised form: 15 Jul 2022; Accepted: 22 Jul 2022; Available online: 27 Jul 2022

Abstract*— *The concepts of connectedness and countability in digital image processing are used for establishing boundaries of objects and components of regions in an image. The purpose of this paper is to investigate some notions of connectedness and countability of Khalimsky line topology.

***Keywords*— Countability, Khalimsky line, Khalimsky arc connected space.**

I. INTRODUCTION

Digital topology has been developed to address problems in image processing, an area of computer science that deals with the analysis and manipulation of pictures by computers. Digital topology provides a sound mathematical basis for image processing operations such as object counting, boundary detection, data compression, and thinning. The basic building block of digital n-space is the digital line or the Khalimsky line. To define a topology on the digital plane, we first consider a topology on integers. This topology can be defined in terms of the minimal neighborhood N(x) of each point x, which is known as Khalimsky topology. Nowadays, this topology is one of the most important concepts of digital topology. The digital line, the digital plane, the three dimensional digital spaces are of great importance in the study of point set theory to computer graphics. In [3] the author combines the one dimensional connectedness of intervals of reals with a point-by-point to construct algorithms that serves as the foundation for digital topology. Abd El-Momen et.al in [6] have shown that the Khalimsky line(digital line) is a typical example of $\mu_{r,s-T_{\frac{1}{2}}}$ spaces. Connectedness and continuity of digital spaces with Khalimsky topology have been discussed in [3],[7],[9]. G.Guterre introduced three definitions of first countable space in [8]. In this paper, some notions of countability and connectedness of Khalimsky line topology are investigated.

II. BASIC CONCEPTS OF GENERAL TOPOLOGY

The foundation of topology is the classical set theory. A topological space is a set X along with a topology τ define on it. A topology on a set is the collection τ of the subsets of the set X such that τ contains the empty set, the set itself, and which is closed finite intersection and arbitrary unions. The elements of this collection are called open sets. Then the ordered pair (X, τ) is termed a topological space. We generally find a basis to generate topology on a set. [2]

Definition: Let (X, τ) be a topological space. Let $\mathcal{B}$ be a class of open subsets of X, i.e. $\mathcal{B} \subset \tau$. Then $\mathcal{B}$ is a base for the topology τ iff (i) every open set $G \in \tau$ is the union of members of $\mathcal{B}$. Equivalently, $\mathcal{B} \subset \tau$ is a base for τ

iff (ii) for any point p belonging to an open set G, there exists $B \in \mathcal{B}$ with $p \in B \subset G$. [5]

The open intervals from a base for the usual topology on the real line$\mathbb{R}$. For if $G \subset \mathbb{R}$ is open and $p \in G$, then by definition, there exists an open interval (a,b) with $p \in (a, b) \subset G$.

Local base: Given a point $x \in X$, a family $\mathcal{B}_x \subset \tau$ is a local base at x if for every neighborhood U of x, there exists $B \in \mathcal{B}_x$ such that $B \subset U$.

Covers: Let $C = \{G_i\}$ be a class of subsets of X such that $A \subset \cup G_i$ for some $A \subset$ X. Then C is called a cover of A, and an open cover if each G_i is open. If C contains a countable (finite) subclass which is also a cover of A, then C is said to be countable cover or C is said to contain a countable subcover.

Closed set: Let X be a topological space. A subset A of X is a closed set iff its complements A^c is an open set.

Closer of a set: Let A be a subset of a topological space X. The closure of A, denoted by A^- is the intersection of all closed supersets of A, i.e if $\{F_i : i \in I\}$ is the class of all closed subsets of X containing A, then $A^- = \cap_i F_i$

Interior: Let B be a subset of a topological space X. A point $p \in B$ is called an interior point of B if p belongs to an open set G contained in B: $p \in G \subset B$ where G is open

The set of interior points of B, denoted by int(B) is called the interior of B.

Exterior: The exterior of B written ext(B), is the interior of the complement of B, i.e int(B^c).

The boundary of B, written b(B), is the set of points which do not belong to the interior or the exterior of B.

Neighborhoods and Neighborhood system:

Let p is a point in any topological space X. A subset N of X is a neighborhood of p iff N is a superset of an open set G containing p: $p \in G \subset N$ where G is an open set.

The class of all neighborhoods of p$\in X$, denoted by N_p, is called the neighborhood system of p.

Arc wise connected sets: A subset E of a topological space X is said to be arcwise connected if for any two points x,y $\in$ E there is a path f: I $\rightarrow$X from x to y which is contained in E. [5]

2.1 Topological structure of digital images

The notion of a topological structure provides a setting for the analysis of digital images. Let X denote a set of picture points (picture elements) in a digital image. A topological structure on a set X is a structure given by a set of subsets τ of X, which has the following properties

(i) Every union of sets in τ is a set in τ

(ii) Every finite intersection of sets in τ is a set in τ

A digital image topological space is a digital image equipped with a topological structure.

III. THE KHALIMSKY LINE

The Khalimsky line is the integers, $\mathbb{Z}$, equipped with the topology κ generated by $G_\kappa = \{2n-1, 2n, 2n+1\}$. Two integers x,y are adjacent if $|x-y| = 1$.

A subset I of $\mathbb{Z}$ is an interval (of integers) if whenever $x, y \in I$ and $x < z < y$, then $Z \in I$.

Proposition 3.1: A subset of $\mathbb{Z}$ is open iff whenever it contains an even integer, it also contains its adjacent integers. It is closed iff whenever it contains an odd integer, it also contains its adjacent integers.[3]

Corollary 3.2: The connected components of a set of integers are the maximal intervals it contains. A set of integers is connected iff it is an interval.[3]

IV. COUNTABILITY OF KHALIMSKY LINE TOPOLOGY

A countable set is a set with the same number of elements as a subset of the set of natural numbers. For example, the set of picture points in a digital image is countable. [10].

Definition**:** A topological space X is said to be first countable if every $x \in X$ has a countable local base and X is said to be second countable if it has a countable basis. [5], [8]

Proposition 4.1: Let $(\mathbb{Z}, \kappa)$ be a topological space where κ is a topology generated by $G_\kappa = \{2n-1, 2n, 2n+1\}$. Then$(\mathbb{Z}, \kappa)$ is second countable.

Proof: Let U be the class of open subsets in $\mathbb{Z}$. Then U is countable set as $\mathbb{Z}$ is countable, and furthermore, is a base for the topology on $\mathbb{Z}$. Hence $(\mathbb{Z}, \kappa)$ is second countable space.

Proposition 4.2: Let A be any subset of a second countable space $(\mathbb{Z}, \kappa)$. If C is an open cover of A, then C is reducible to a countable cover.

Proof: Let $\mathcal{B}$ be a countable base for $(\mathbb{Z}, \kappa)$. Since $A \subset \cup \{G : G \in C\}$, for every $p \in A$,$\exists G_p \in C$ such that $p \in G_p$. Since $\mathcal{B}$ is a base for $(\mathbb{Z}, \kappa)$, for every $p \in A$ $\exists B_p \in \mathcal{B}$ such that $p \in B_p \subset G_p$. Hence $A \subset \cup \{B_p : p \in A\}$. But $\{B_p : p \in A\} \subset \mathcal{B}$, so it is countable, hence $\{B_p : p \in A\} = \{B_n : n \in N\}$, where n is a countable index set. For each $n \in N$ choose one set $G_n \in C$ such that $B_n \subset G_n$. Then

$A \subset \cup \{B_n: n \in N\} \subset \cup \{G_n: n \in N\}$ and so $\{G_n: n \in N\}$ is a countable subcover of C.

V. CONNECTEDNESS OF KHALIMSKY LINE TOPOLOGY

To represent continuous geometrical objects in the computer, notion of connectedness on discrete sets are useful to represent discrete objects.

Definition: Let A be a subset of a topological space (X,τ). Then A is connected with respect to τ if and only if A is connected with respect to the relative topology τ_A on A.

Theorem 5.1: A topological space X is connected if and only if (i) X is not the union of two non-empty disjoint open sets, or equivalently (ii) X and $\emptyset$ are the only subsets of X which are both open and closed.

The real line $\mathbb{R}$ with the usual topology is connected since $\mathbb{R}$ and $\emptyset$ are the only subsets of $\mathbb{R}$ which are both open and closed.

Definition: A connected ordered topological space (COTS), is a connected space X such that if $Y \subseteq X$ contains at least three distinct points, then there is a $y \in Y$ such that $Y - \{y\}$ meets more than one component $X - \{y\}$.

Proposition 5.2: The connected components of a set of real numbers are the maximal intervals it contains. A set of real numbers is connected if it is an interval.

Theorem 5.3: Each COTS X admits a total order $<$ such that for each $x \in X$ the components of $X - \{x\}$ are $\downarrow (x) = \{y | y < x\}$ and $\uparrow (x) = \{y | y > x\}$.

Definition: A topological space is Alexandroff iff arbitrary intersections of open sets are open.

Lemma 5.4: A topological space is Alexandroff iff each element, x, is in a smallest open sets and this set is $n(x)$.

Theorem: Each interval in the Khalimsky line is a locally finite COTS. [3]

Proof: If x is an even integer, then by proposition 3.1 $\{x\}$ is open, and $\{x-1, x, x+1\}$ is closed. If x is odd , then similarly, $\{x\}$ is closed and $\{x-1, x, x+1\}$ is open. Thus each $\{x\}$ is contained in a finite open set and a finite closed set, so $\mathbb{Z}$ is locally finite.

Definition: A digital space is a pair (V, π) where V is a non-empty set and π is a binary symmetric relation on V such that for any two elements x and y of V there is a finite sequence $(x^0, x^1, \dots \dots, x^n)$ of elements in V such that $x = x^0$ and $y = x^n$ and $(x^j, x^{j+1}) \in \pi$ for j = 0,1, ……(n-1).

Definition: A topological space X is said to be Khalimsky arc connected if it satisfies the following conditions:

(i) X satisfies T_0 −axiom

(ii) for all $x, y \in X$, I = $[a,b]_z$ and $\varphi: I \to X$ such that $\varphi(a) = x, \varphi(b) = y$ and φ is a homeomorphism of I into $\varphi(I)$. [7]

Proposition 5.5: Continuous image of Khalimsky arc connected sets are arc connected.

Proof: Let $E \subset X$ be arc connected and let $f: X \to Y$ be continuous. Let $f(E)$ be the image of arc connected space. We have to prove that $f(E)$ is arc connected. Let $p, q \in f(E)$. Then $\exists\, c, d$ such that $f(c) = p$ and $f(d) = q$. But E is arc connected and so there exists a path $\varphi: I \to X$ such that $\varphi(a) = c\varphi(b) = d$ and $\varphi(I) \subset E$. Composition of continuous function is continuous, so $f \circ g: I \to Y$ is continuous. Furthermore, $f \circ \varphi(a) = f(c) = p$, $f \circ \varphi(d) = f(d) = q$ and $f \circ g(I) = f[g[I]] \subset f(E)$.Thus $f(E)$ is Khalimsky arc connected.

VI. CONCLUSION

In this paper countability and arc connectedness of Khalimsky line topology are observed and it is shown that continuous image of Khalimsky arc connected sets are arc connected. Some properties regarding connectivity which is the particular interest in image processing have been studied. In future, I will study continuity and quasi seperablity of the same topology.

REFERENCES

[1] A. Rosenfeld, Digital topology, Amer. Math. Monthly, 86 (1979), 621 – 630

[2] S.Mishra and M.Aaliya, Application of Topology in Science and Technology, International Journal of Research and Analytical Reciews,5(2018)101-104.

[3] Kopperman, R. (1994). The Khalimsky Line as a Foundation for Digital Topology. In: O, YL., Toet, A., Foster, D., Heijmans, H.J.A.M., Meer, P. (eds) Shape in Picture. NATO ASI Series, vol 126. Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-662-03039-4_2 .

[4] S. Jafari and A. Selvakumar, On Some Sets in Digital Topology, Poincare Journal of Analysis & ApplicationsVol. 8, No. 1(I) (2021), DOI: 10.33786/pjaa.2021.v08i01(i).001.

[5] Seymour Lipschutz, General Topology, Schaum's Outline Series.

[6] Abd El-Monem M. Kozae1 and El-Sayed A. Abo-Tabl, On Digital Line and Operations, General Letters in Mathematics Vol. 4, No. 3, June 2018, pp.107-113, https://doi.org/10.31559/glm2018.4.

[7] Anne Kurie K., M. S. Samuel,Continuity in Digital Spaces with The Khalimsny Topology,International Journal of MathematicsTrends and Technology 53(2018),65-67

[8] G.Gutierres,What is First Countable Space?, Topology and Its Applications,153(2006)3420-3429. doi:10.1016/j.topol.2006.03.003

[9] K. Annie Kurien1 and M. S. Samuel, Connectedness in Digital Spaces with the Khalimsky Topology, Int. J. Math. And Appl., 6(1–D)(2018), 773–774.

[10] James F. Peters, Topology of Digital Images, Springer-Verleg Berlin Heidelkberg,2014.