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Existence and Uniqueness Result for a Class of Impulsive Delay Differential Equations

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Abstract— *In this paper, we investigate the existence and uniqueness of solutions of the formulated problem of impulsive delay differential equation with continuous delay. The strategy adopted for the proof is based on Caratheodory's techniquesand Lipschitz condition is required to obtain uniqueness.*
Keywords— Impulsive, delay differential equation, existence, uniqueness.

I. INTRODUCTION

From late1980s, research in impulsive delay differential equations has been undergoing some exciting growth or moment. This to a large extent can be attributed to the quest by mathematicians in particular and the science community in general to unveil nature the way it truly is. The realization that differential equations in general and indeed impulsive delay differential equations are very important models for describing the true state of several real life processes/phenomena may have been the tunic. One can attest that in most human processes or natural phenomena, the present state are most often affected significantly by their past state and those we thought of as continuous may indeed undergo abrupt change at several points or even be stochastic. Thus, research search light is gradually being turned on processes or phenomena which exhibit both of these traits which were probably avoided due to technical difficulties. In fact, new applications for this class of differential equations continue to arise with increasing rapidity in modelling processes in field as devise as physics, engineering, technology, control theory, industrial robotics, economics, environmental sciences, ecology, medicine, just to mention a few ([1], [2], [3], [4], [5]). Recent works in the area can be seen in ([6], [7], [8], [9], [10])

It is worthwhile to state that unlike other areas in differential equations, impulsive delay differential equations are relatively at their early stages of development and so a lot still need to be done so as to have full grasp of the various qualitative properties and their applications ([11], [12], [13], [14]). Evidence that the development of theories and basic concepts in impulsive differential equations with delays are not yet settled can be found in the nearly absence or draft of texts in this area as can be seen in areas such as ODE and delay or impulsive differential equations. This work aims at projecting some crucial issues which are involved in formulating impulsive delay differential equations (especially those with continuous delays), existence/uniqueness of solution and their implication to the wider community of scientists and the world in general. We will present some earlier developments in this area with a view to harmonizing them with our findings and hope that our research audience will be exposed to a new frontier of knowledge and we sincerely believe that the world will be the better for it in the years ahead.

II. CLASSICAL EXISTENCE RESULTS

Before delving into the main results of this work, it is necessary to quote some very useful classical results that will illuminate some of our findings.

Definition 2.1 ***(Solution of an initial value problem)*** Let $\Omega \subset R^n \times R$ be an open set and let $f:\Omega \subset R^n \times R \to R^n$ be a continuous function. Let $(\xi, t_o) \in \Omega$ and let $B_\varepsilon(\xi) \times (t_0 - \sigma, t_0 + \sigma) \subset \Omega$ be a spherical neighbourhood. *A continuously differentiable function* $\varphi : [t_0, t_0 + \sigma) \to B_\varepsilon(\xi)$ *is a solution of the initial value problem*

$$\begin{cases} \dfrac{dx(t)}{dt} = f\left(x(t), t\right) \\ x(t_0) = \xi \end{cases} \tag{2.1}$$

on the interval $[t_0, t_0 + \sigma)$ if φ satisfies the system of equations

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$$\begin{cases} \dfrac{d\varphi(t)}{dt} = f\left(\varphi(t),t\right) \forall t, t_0 \le t < t_0 + \sigma \\ \varphi(t_0) = \xi. \end{cases} \tag{2.2}$$

This is one of the simplest formulations of the initial value problem. It is important that a solution of the initial value problem (2.1) must fulfil an initial value restriction and the derivative must satisfy the equation on an interval containing the initial point.

Theorem 2.1*(Cauchy Existence Theorem):*

If $f:\Omega \subset R^n \times R \to R^n$ is a continuous function as described above on an open set Ω then for any $(\xi, t_o) \in \Omega$ there exists an open spherical neighbourhood $B_\varepsilon(\xi) \times (t_0 - \sigma, t_0 + \sigma) \subset \Omega$ such that the initial value problem (2.1) has a solution $\varphi: [t_0, t_0 + \omega) \to B_\varepsilon(\xi)$ for $\sigma > \omega > 0$ ([15], [16], [17]).

Remark 2.1 It is elementary knowledge in this field that the solutions of such equations are not unique. If we add the local Lipschitz property at each point of Ω then uniqueness follows.

Theorem 2.2*Existence and Uniqueness (Piccard-Lindelöf):*

If $f:\Omega \subset R^n \times R \to R^n$ is a continuous function and fulfils local Lipschitz condition in x, for each fixed t on the open set Ω then for any $(\xi, t_o) \in \Omega$ there exists an open spherical neighbourhood $B_\varepsilon(\xi) \times (t_0 - \sigma, t_0 + \sigma) \subset \Omega$ such that the initial value problem (2.1) has a unique solution $\varphi: [t_0, t_0 + \omega) \to B_\varepsilon(\xi)$ for $\sigma > \omega > 0$ ([15], [16], [17]).

Control theory brought out the need first to prove the existence of initial value problems for non-continuous right sides which equivalently sought the solutions in the category of absolute continuous functions. Introduction of the extended solutions of partial differential equations by using Sobolev spaces have manifested such a jump.

Our impulsive differential equations already operate with absolute continuous solutions. Introduction of delays simply changes the solution to absolute continuous equations since the right sides are measurable and not piecewise continuous functions. We will need Caratheodory's result which is the strongest of its type that made it possible to handle control theory. We hope it will help us in our discussion.

Theorem 2.3*(Caratheodory):* *Let $f:\Omega \subset R^n \times R \to R^n$ be a continuous function in x for each fixed t and let it be measurable in t for each fixed x, $\forall$ (x, t) $\in$ Ω. Moreover, for any $(\xi, t_o) \in \Omega$ there exists an open spherical neighbourhood $B_\varepsilon(\xi) \times (t_0 - \sigma, t_0 + \sigma) \subset \Omega$ and a Lebesgue integrable function m such that $m:(t_0 - \sigma, t_0 + \sigma) \to R^+$ and the inequality $\| f(x,t) \| \le m(t), \forall (x,t) \in B_\varepsilon(\xi) \times (\theta - \sigma, \theta + \sigma)$ holds. Then the initial value problem (2.1) has a solution $\varphi: [t_0, t_0 + \omega) \to B_\varepsilon(\xi)$ for $\sigma > \omega > 0$ in generalised sense* ([15], [16], [17]).

Remark 2.2 Under the conditions formulated in Theorem 2.3, the solution of the equations is not unique. If we add local Lipschitz property in x for each fixed t at each point of Ω then the uniqueness follows. Theorems on minimal, maximal solutions and conditions of uniqueness can be found in [15] .

Condition 2.1 Let $\psi:(0,a) \times (0,\infty) \to R^+$ be a continuous function, non-decreasing in the second parameter for any fixed value of the first one. The function ψ is a Lipschitz function if $\gamma(t) := 0, \forall t, 0 \le t < \alpha$ and for some $0 \le \alpha < a$ is the only differentiable function on [0, α) which fulfils the conditions

$$\gamma_+'(0) = \lim_{t \to 0+} \frac{\gamma(t) - \gamma(0)}{t} \text{ exists;} \tag{2.3}$$

$$\gamma'(0) = \psi(t, \gamma(t)), \forall t, 0 \le t < \alpha \tag{2.4}$$

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$$\gamma(0)=\gamma_{+'}(0)=0 \tag{2.5}$$

Remark 2.3 If ψ (t, r) = L × r then ψ satisfies the conditions and then we will have Lipschitz condition in (2.3).

Theorem 2.4 *Let* $\psi:(0,a)\times(0,\infty)\to R^+$ *be a continuous function that fulfils Condition 2.1 and let* $f:\Omega\subset R^n\times R\to R^n$ *be a continuous function in x for each fixed t and let it be measurable in t for each fixed x* $\forall(x,t)\in\Omega$. *Moreover,* $\forall(\xi,\theta)\in\Omega$ *there exists an open spherical neighbourhood* $B_\varepsilon(\xi)\times(\theta-\delta,\theta+\delta)\subset\Omega$ *such that*

$$\|f(x,t)-f(y,t)\|\le\psi(|t-\theta|,\|x-y\|)\ \forall(x,t),(y,t)\in B_\varepsilon(\xi)\times(\theta-\delta,\theta+\delta),t\neq\theta. \tag{2.6}$$

Moreover there exists a non-negative Lebesgue integrable function $m_{\theta,\delta}:(\theta-\delta,\theta+\delta)\to R^+$ such that

$$\psi(t,r)\le m_{\theta,\delta},(t,r)\in(\theta-\delta,\theta+\delta)\times(0,\varepsilon). \tag{2.7}$$

Then the initial value problem (2.1) has a unique solution $\varphi:[\theta,\theta+\beta)\to B_\varepsilon(\xi)$ for $\delta>\beta>0$ in generalised sense.

Remark 2.4 This theorem is a combination of the theorems 1.1 and 2.1 or 2.2 in [15].

Delayed Impulsive Differential Equations

With some of the major classical existence results in view, we now turn our attention to what the situation will be in delayed impulsive equations. We will use the notations introduced in earlier sections to simplify the very complex notations of impulsive delay differential equations in what follows. Prior to discussing the major concepts, we put forward one of the underlying assumptions.

Assumption 2.1 *Let* $h_j:T\to R^+\forall j,1\le j\le m$ *be continuous delay functions that fulfil conditions in remark 3.5a and 3.5b. Also, let* $F:R^{(m+1)\times n}\times T\to R^n$ *and* $F^*:PC[T,R^n]\times S\to R^n$ *be continuous functions.*

We introduce the following notations:

$$r^{[j]}:=\sup_{s\in(t_j,t_{j+1}]}\max_{1\le k\le m}h_k(s)\ \ge\ r_{[j]}:=\inf_{s\in(t_j,t_{j+1}]}\min_{1\le k\le m}r_k(s)>0. \tag{2.8}$$

Remark 2.5 *If the delays are constants (* h_j *are independent of time), say* $h_j=\tau_j,\forall 1\le j\le m$ *are constants, then we can modify equation (2.8) as follows:*

$$r^{[j]}:=\max_{1\le k\le m}h_k\ge r_{[j]}:=\min_{1\le k\le m}h_k>0. \tag{2.9}$$

$$r^{[j]}:=\max_{1\le k\le m}\tau_k\ge r_{[j]}:=\min_{1\le k\le m}\tau_k>0 \tag{2.10}$$

We are now in a position to formulate the problem of delayed impulsive differential equations. An initial value problem for impulsive differential equation with delay is formulated normally as follows:

III. RESULTS

Formulation of Initial Value Problem for Delayed Impulsive Differential Equations

Let $h_j:T\to R^+,1\le j\le m$ be continuous delay functions, let $F:R^{(m+1)\times n}\times T\to R^n$ and $F^*:PC[T,R^n]\times S\to R^n$ be continuous functions. Let $\theta\in T$ and let $t_j\in S$ be the unique impulse point which satisfies $t_j\le\theta<t_{j+1}$. Let $r^{[j]}$ be defined as in (2.9) and let $(\xi,\theta)\in PA[[\theta-r^{[j]},\theta],R^n]\times\{\theta\}\subset PA[[\theta-r^{[j]},\theta],R^n]\times[t_j,t_{j+1})$.

Definition 3.1 The delayed impulsive initial function problem with initial function and time $(\xi,\theta)\in PA[[\theta-r^{[j]},\theta],R^n]\times\{\theta\}\subset PA[[\theta-r^{[j]},\theta],R^n]\times\{[t_j,t_{j+1})\}$ is to find a function $x\in PA[[\theta-r^{[j]},\theta+\alpha],R^n],0<\alpha<t_{j+1}$ that satisfies the conditions

$$\frac{dx(t)}{dt}=F\circ(\hat{x}\circ(t-\bar{h}\circ\hat{t},x(t),t))\ \forall\ t\in[\theta,\theta+\alpha) \quad (3.1)$$

$$x(s)=\xi(s),\forall s\in[\theta-r^{[j]},\theta] \quad (3.2)$$

Theorem 3.1 *Suppose* $r^{[j]}$ *and* $r_{[j]}$ *are defined as in (2.9),(2.10) and* $(\xi,\theta)\in PA[[\theta-r^{[j]},\theta],R^n]\times[t_j,t_{j+1}$ *is given. If F and* F^* *are continuous functions with* $\gamma:=min\{\theta+r_{[j]},t_{j+1},\ \alpha\}$, *then* $\forall(y,t)\in R^n\times[\theta,\theta+\gamma)$,

$$\Phi(y,t):=F\circ(\xi\circ(\hat{t}-\bar{h}\circ\hat{t},y,t))=F(\xi(t-h_1(t)),(\xi(t-h_2(t)),\cdots(\xi(t-h_m(t)),y,t). \quad (3.3)$$

This is a well defined function which is continuous in y for all fixed $t\in[\theta,\theta+\gamma)$ and it is measurable in t for all $y\in R^n$. Moreover, if F fulfils global Lipschitz condition in its spatial coordinates then $\Phi(y,t)$ also fulfils global Lipschitz condition in y for each fixed t.

Proof. **Existence:** Since $t\in[\theta,\theta+\alpha)$ it follows that $\theta\le t<\theta+r_{[j]}$, and

$$\theta-r^{[j]}\le t-h_p(t)\le\theta-r_{[j]}<\theta;1\le p\le m. \quad (3.4)$$

Hence, Φ and the parameters in its definition are well defined. We know from the definition of F that Φ is continuous in y for each fixed t, and if F is Lipschitzian in the spatial co-ordinates for each fixed t then Φ is also Lipschitzian. This proves the theorem.

Theorem 3.2 *Let* $r^{[j]},r_{[j]}$ *be defined by (2.8),* $(\xi,\theta)\in PA[[\theta-r^{[j]},\theta],R^n]\times[t_j,t_{j+1})$ be given and let $\gamma:=min\{\ \theta+r_{[j]},t_{j+1}\}$. Let $F:R^{(m+1)n}\times T\to R^n$ and $F^*:PC[T,Rn]\times S\to R^n$ be continuous functions and satisfy a global Lipschitz condition. Then $\exists\delta>0$ such that $\gamma\ge\delta+\theta>0$ and a unique $x\in PA[[\theta-r^{[j]},\theta+\alpha],R^n]$ satisfying

$$\begin{aligned} &x'(t)=\Phi(x(t),t):=F\circ(\hat{\xi}\circ(\hat{t}-\bar{h}\circ\hat{t}),x(t),t))\ \forall t\in[\theta,\theta+\delta];\\ &x(t)=\xi(t)\forall t\in[\theta-r^{[j]},\theta] \end{aligned} \quad (3.5)$$

Proof. Let

$$M_{[\theta-\tau^{[J]},\theta]}=2\cdot\|\xi\|_{[\theta-r^{[J]},\theta]} \quad (3.6)$$

let

$$B_{\xi,\theta}:=\left(B_{M_{[\theta-r^{[j]},\theta]}}\left(\xi(0)\right)\right)\times\left[\theta-r^{[j]},t_{j+1}\right] \quad (3.7)$$

and let

$$m_{[\theta-r^{[j]},t_{j+1}]}:=\sup_{(y,t)\in B_{\xi,\theta}}\|\Phi(y,t)\|. \quad (3.8)$$

Then the constant function $m_{[\theta-r^{[j]},t_{j+1}]}\ge\|\Phi(y,t)\|,\forall(y,t)\in B_{\xi,\theta}$ and it is integrable on $[\theta-r^{[j]},t_{j+1}]$. Hence by

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Caratheodory's theorem, there exists a solution on the interval $[\theta - r^{[j]}, \theta + \delta]$ with $\delta := min\{\gamma, \frac{M_{[\theta - r^{[j]}, \theta]} - \|\xi(\theta)\|}{m_{[\theta - r^{[j]}, t_{j+1}]}}\}$. By Lipschitz condition, the solution is unique.

Theorem 3.3 *Suppose* $r^{[j]}$ *and* $r_{[j]}$ *are defined by (2.8) and* $(\xi, \theta) \in PA[[\theta - r^{[j]}, \theta], R^n] \times \{[t_j, t_{j+1})$ is given.

Let $\gamma := min\{\theta + r_{[j]}, t_{j+1}\}$ and let F and F^* fulfil assumption 4.1 and satisfy a global Lipschitz condition. Using the definitions (3.6) and (3.7) let

$$\delta := min\left\{t_{j+1}, \frac{M_{[\theta - r^{[j]}, \theta]} - \|\xi(\theta)\|}{m_{[\theta - r^{[j]}, t_{j+1}]}}\right\} > 0. \tag{3.9}$$

Then a unique solution $x \in PA[[\theta - r^{[j]}, \theta + \alpha], R^n]$ of (3.5) exists for $\delta > 0$ defined by (3.9) that satisfies

$$\begin{aligned} &x'(t) = \Phi(x(t), t) := F \circ (\hat{x} \circ (\hat{t} - \bar{h} \circ \hat{t}), x(t), t)) \ \forall t \in [\theta, \theta + \delta]; \\ &x(t) = \xi(t) \ \forall t \in [\theta - r^{[j]}, \theta]. \end{aligned} \tag{3.10}$$

Proof. Note that in Theorem 3.2 the interval for the existence of solution is $[\theta, \theta + \delta]$ where

$$\begin{aligned} \delta &:= min\left\{\gamma, \frac{M_{[\theta - r^{[j]}, \theta]} - \|\xi(\theta)\|}{m_{[\theta - r^{[j]}, t_{j+1}]}}\right\} = \\ &= min\left\{\theta + r_{[j]}, t_{j+1}, \frac{M_{[\theta - r^{[j]}, \theta]} - \|\xi(\theta)\|}{m_{[\theta - r^{[j]}, t_{j+1}]}}\right\} \end{aligned} \tag{3.11}$$

which means that the length depends on the smallest size of the delay. In (3.5) all quantities are defined over the interval $[\theta - r_{[j]}, t_{j+1}]$ as formulae (3.7) and (3.8) show.Therefore the standard extension technique applies and after a finite number of steps, the solution is extended to $[\theta, \theta + \delta]$ in $r^{[j]}$ size steps.

IV. CONCLUSION

Here, we have been able to formulate a problem of impulsive delay differential equation with continuous delays as seen in section 3. The strategy adopted for the existence and uniqueness proof is based on Caratheodory's techniques. The function $f(t, x(t), x(t - h(t))$ is considered to be a generalised function with derivatives almost everywhere in $PA[T, R^n]$, continuous in x for fixed t and measurable in t for each fixed x. To obtain uniqueness, we required that Lipschitzcondition be fulfilled.

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